Essays on Competition and Returns
of Inside Money

Tese submetida à Escola de Pós-Graduação em Economia da
Fundação Getulio Vargas como requisito para obtenção do
Título de Doutor em Economia

Aluno: Henrique Dezemone Forno

Professor Orientador: Ricardo de Oliveira Cavalcanti

Rio de Janeiro
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Aluno: Henrique Dezemone Forno

Banca Examinadora:
Professor Ricardo de Oliveira Cavalcanti (Orientador, EPGE/FGV)
Professor Aloisio Pessoa de Araújo (EPGE/FGV)
Professor Arilton Teixeira (IBMEC/RJ)
Professor Eurilton Alves Araújo Júnior (IBMEC/RJ)
Professor Paulo Klinger Moreira (EPGE/FGV)

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Abstract

In an early paper, Cavalcanti and Wallace (2001) showed, using a computable version of Cavalcanti-Wallace model (CW-1999), that optimal regulation induces banks to pay interests, instead of contracting the money supply in an inside money allocation.

Here, we generalize CW in two fashions, assuming inside money allocations, so that banks are supposed to issue money as they find a potential producer wishing to produce. The first generalization allows for seasonality due to real shocks on preferences with persistence and for monetary policy improvement. We found an asymmetric path for interest rates when constraints matter, even when shocks are independent. The second generalization allows for bank competition, in the sense that banks can choose between two different banking nets. We proof the existence of simple stable and unstable equilibria and also verify the existence of multiple equilibria.

*JEL Classification Codes: C61; D82 ;E40 and E50.*
1 Introduction

One of the oldest principles that guide monetary policy decisions in terms of economic stability is that money needs to bear a satisfactory return. As the use of payments is unavoidable in modern economies, when money bears insufficient earns, economic agents take suboptimal decisions, because they save in the use of money. When that happens, real product level turns out to be less than it would be if money earned more.

As stated by Milton Friedman and other researchers, central bank should run a deflation as an attempt to increase the return of money. This kind of proposition has been used to guide monetary policy worldwide. However, microeconomics has not been developing fast enough to support this idea.

The optimality of deflation has been derived from agent representative models, where the optimal rate of interest depends exclusively on the time discount factor. These kinds of models have been criticized, because is not possible to describe the social benefit of money without making references to the trades provided by it. But, if the model has representative agents, there must not exist any benefit derived from trade.

Monetary models with trades are quite recent. The most known example is due to Kiyotaki and Wright (1989), where money functions as a medium of exchange. Usual aggregation techniques do not work in this case. An equilibrium is described by a distribution of currency over heterogeneous agents, in order to make money desirable by preserving its scarcity.

In principle, deflation could be a robust way to assure the value of money. However, in modern times, most payments are provided by private agents (check cards, credit cards, checks, smart cards, e-cash, etc).

Cavalcanti-Wallace’s model offers a revision of the Kiyotaki-Wright’s model that allows for private payments. Theirs is a random-matching model of money in which some people, called bankers, can be monitored via a record-keeping technology while others, called nonbankers, cannot be monitored, because their trading histories are private information. Would the proposition about the optimality of deflation be preserved here?

Cavalcanti and Wallace (2001) showed, using a computable version, that optimal regulation induces banks to pay interests, instead of contracting the money supply, in an inside money allocation.

As stated by the authors, private payments are a permanent trend in modern economies and central banks must be able to control the money supply offered by banks, as the last use modern technologies each
day easier to track. Moreover, monetary policy would increasingly focus on the provision of bank services in more elaborate ways than the simple control of monetary aggregates.

Here, we generalize the Cavalcanti-Wallace model (CW) in two fashions, assuming inside money allocations, so that banks are supposed to issue money as they find a potential producer wishing to produce.

The first generalization, Cavalcanti and Forno’s model, henceforth CF, allows for seasonality due to real shocks on preferences with persistence and for monetary policy improvement.

CF is a random-matching banking model, with asymmetric monitoring and absence of double coincidence of wants. Optimum allocations can be unconstrained, in the sense that the quantity of output that the public trades for money is invariant to small changes in the discount factor, or in the size of the bank sector.

Using the representative agent welfare criteria, our purpose is to characterize the optimal regulation of banks, when the value of money is restricted by the public participation constraint in all period states due to real shocks. From CW we know that banks increase return on money in order to improve public participation, i.e., payment of interest on inside money is an effective mechanism. In CF we want to know how banks react to real shocks with different levels of persistence. Moreover, we should ask ourselves if it is possible for monetary policy maker to smooth production, and as a result, consumption.

Money-issuing banks arise endogenously in the analysis, as an institution best using the economy’s capacity to monitor a limited measure of individuals, called bankers. The remaining individuals, the nonbank public, become the users of the inside money provided by banks.

For a large region of the parameter space, nonbank output is restricted by the incentives to acquire fiat objects, while bank output is not, since monitoring facilitates credit and creates, therefore, higher incentives to bankers.

A visible shortcoming of monetary models is the lack of predictions concerning the valuation of coexisting private money. Therefore, the second generalization allows for bank competition, in the sense that banks can choose between two different banking nets, which issues different notes and do not cooperate.

Cavalcanti (2000) discussed the gains from having a multiplicity of currencies, differentiated by something like their colors, on a random-matching model.

The analysis here focus on the case when constraints do not bind at first-best. This simple case suffices
to reach the basic economic ideas we intend to show.

The rest of this thesis is organized as follows. In chapter 2, we present the environment of CF and implementable, stationary and symmetric inside money allocations we will work with, namely simple implementable allocations. In chapter 3, we consider CF with no persistence and no monetary policy improvement. CW is a particular case where there are no real shocks. This procedure supplies good test cases for chapter 4, which allows both for persistence and for monetary policy role. In chapter 4 we also explain the guidelines and cite the tools used in computations to solve the restricted maximization problem. In chapter 5, we modify again CW in order to derive, in a monetary model, a banking sector concentration scheme. In chapter 6, we conclude.
2 The economy

2.1 Cavalcanti and Forno model (CF)

In this chapter, we generalize the Cavalcanti-Wallace’s model (CW), assuming inside money allocations\(^1\), so that banks are supposed to emit money as they find a potential producer wishing to produce. This generalization allows for seasonality due to real shocks on preferences with persistence and for monetary policy improvement.

2.1.1 The environment

Time is discrete and the horizon is finite. There are \(S\) distinct, divisible, and perishable types of goods at each date and there is a continuum of each of \(S\) specialization types of people, where \(S > 2\) to avoid double-coincidence meetings: a person whose specialization type is \(s\) consumes only good \(s\) and produces only good \(s + 1\) (modulo \(S\)), for \(s = 1, 2, ..., S\). We set \(S = 3\). Each person maximizes expected discounted utility with discount factor \(\beta \in (0, 1)\).

Utility in consuming in a period is given by \(u(y_c)\), where \(y_c\) is the amount consumed. Utility in producing in a period is given by \(-y_p\), where \(y_p\) is the amount produced\(^2\).

The function \(u(y_c)\) is defined on \([0, \infty)\), is increasing, twice differentiable, strictly concave, and satisfies \(u(0) = 0\). Moreover, there exist \(\tilde{y} > 0\) such that \(u(\tilde{y}) = \tilde{y}\).

There are two period states, namely low and high, in which preferences differ. We use a Cavalcanti-Wallace-based specification\(^3\), i.e., a constant absolute-risk-aversion one:

\[
u_j(y_c) = 1 - e^{-A_jy_c},
\]

where \(A_j = e^{\gamma_j}\) for \(j \in \{l\ (low), h\ (high)\}\).

It can be easily verified that first-best level of output in period state \(j\), \(y^*_j\); the maximizer of \(u_j(y_c) - y_p\), equals \(\frac{\log(A_j)}{A_j}\).

\(^1\)In an earlier paper, Cavalcanti and Wallace (1999) showed that the set of implementable outcomes using outside money is a strict subset of the set using inside money.

\(^2\)Note that this assumptions are made only for simplicity. In the next chapters, it will be made clear that we can relax them.

\(^3\)Cavalcanti and Wallace (2001).
We set $\gamma_l = 3.00$ and $\gamma_h = 2.50^4$. We have simulated other values and the results did not change qualitatively.

In each period, people are randomly matched in pairs. There are two kinds of meetings: single-coincidence meetings, those between a type $s$ person (the producer) and a type $s + 1$ person (the consumer) for some $s$; and no-coincidence meetings, those in which neither person produces what the other consumes. We assume that people cannot commit to future actions, so that those who produce have to get future reward for doing so.

The society is able to keep a public record of the actions of a fraction $B$ of each type of person, where $B \in [0, 1]$. There is no public record for any one else. As we will see, a person whose history is known, a banker, can be induced to produce without receiving something tangible in exchange, because she can be rewarded in the future for actions they take currently; in contrast, a person whose history is not known, a nonbanker, must receive something tangible in order to produce.

We also assume that banks have a technology that permits them to create indivisible and perfectly durable monetary objects, called notes or inside money, that can be freely destroyed. Outside money consists of notes when notes are never destroyed or created. This is necessarily the case when $B = 0$, but in virtue of the dominance result in CW$^5$, there is creation and destruction of money in an optimum when $B \in (0, 1]$.

To maintain the model simple, we assume that each person can carry from one date to the next at most one unit of money.

As a consequence, banks can improve economic welfare by emitting inside money and modifying the value of the money payback. As you will note further, in an outside money allocation, agents are also able to raise welfare when real shocks take place.

We assume that people in a meeting know each other’s specialization type, asset holdings, and identity in the sense of banker or nonbanker.

Finally, we allow for monetary policy improvement in the sense that policy maker can implement $n_l$ and $n_h$ subsequent subperiods for periods low and high, respectively.

---

$^4$We use this notation, because $y^*_\text{low} < y^*_\text{high}$ for $1 \leq \gamma_{\text{high}} < \gamma_{\text{low}}$.

$^5$Cavalcanti and Wallace (1999).
To be more specific, define $P$ the transition matrix,

$$
\begin{pmatrix}
\Pi_1 & \Pi_2 \\
\Psi_2 & \Psi_1
\end{pmatrix}
= \begin{pmatrix}
0 & \pi & 0 & \cdots & 0 & 0 \\
0 & 0 & \pi & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \pi \\
0 & 0 & 0 & \cdots & 0 & \pi
\end{pmatrix}
$$

where,

$$
\Pi_1 = \begin{pmatrix}
0 & \psi & 0 & \cdots & 0 & 0 \\
0 & 0 & \psi & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \psi \\
0 & 0 & 0 & \cdots & 0 & \psi
\end{pmatrix}, \quad \Pi_2 = \begin{pmatrix}
1 - \pi & 0 & 0 & \cdots & 0 & 0 \\
1 - \pi & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 - \pi & 0 & 0 & \cdots & 0 & 0 \\
1 - \pi & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
$$

$$
\Psi_1 = \begin{pmatrix}
1 - \psi & 0 & 0 & \cdots & 0 & 0 \\
1 - \psi & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 - \psi & 0 & 0 & \cdots & 0 & 0 \\
1 - \psi & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad \Psi_2 = \begin{pmatrix}
0 & \psi & 0 & \cdots & 0 & 0 \\
0 & 0 & \psi & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \psi \\
0 & 0 & 0 & \cdots & 0 & \psi
\end{pmatrix}
$$

When the economy is in state $l$, the probability of changing to first substate of $h$ is $(1 - \pi)$ and the probability of maintaining the state, changing to next substate of $l$, is $\pi$. In the same way, we denote $\psi$ the persistence of state $h$.

### 2.1.2 Stationary and symmetric inside money allocations

An allocation describes what happens in different kinds of meetings. Here we define the set of stationary and symmetric allocations. The set of implementable allocations is a subset of that.

One symmetry we impose is across specialization types. Another is that all notes issued by any bank are treated identically.

We impose stationarity in several senses. At the start of a date, prior to being randomly matched, each
person has a *state*. The *state* for a nonbanker is inside money holdings, a member of the set \( \{0, 1\} \). Since banks have known histories, we assume that banks will be permitted to issue money\(^6\).

Cavalcanti and Wallace (2001) permitted for two possible states: a bank in state 1 would be permitted to issue money and a bank in state 0 would not. Moreover, they treated the distribution of banks across states as endogenous and found optimum measure of banks not allowed to issue to be zero for most parameters of interest or nearly zero.

We also require, as part of stationarity, that the distribution of bankers and the fraction of nonbankers in each state are constant.

The above symmetries and stationarities permit us to describe allocations making reference to what happens in single-coincidence meetings at a particular date. We allow what happens to depend on the identity (banker or nonbanker) and state (0 or 1) of the nonbanker producer or consumer\(^7\). Thus, we let production and consumption in a single-coincidence meeting be denoted \( y^{km}_{ij<p}> \in \mathbb{R}_+ \) where the superscripts denote identity - \( k, m \in \{\text{b(banker)}, \text{n(nonbanker)}\} \) where \( k \), the first superscript, is the identity of the producer and \( l \), the second, is that of the consumer - and the subscripts denote states for nonbankers - \( i, j \in \{0, 1\} \) where \( i \), the first subscript, is that of the producer and \( j \), the second, is that of the consumer - and period state \( <p> \), for \( p \in \{l_1, ..., l_n, h_1, ..., h_n\} \).

We also need to describe the distribution of bankers and the distribution of nonbankers across states. We let \( x^k_i \) with \( k \in \{\text{b,n}\} \) and \( i \in \{0, 1\} \) denote the fraction of each production-consumption specialization type who have identity \( k \) and, in case of nonbankers, who are in state \( i \). So, these fractions satisfy:

\[
x^b_i = B \quad \text{and} \quad x^n_0 + x^n_1 = (1 - B)
\]  

An stationary and symmetric inside money allocation is \( (x^k_i, y^{km}_{ij<p}> \) for \( i \in \{0, 1\} \) and \( k, m \in \{\text{b,n}\} \) that satisfies two conditions. The first, if two nonbankers in a meeting have the same state, then neither can switch to a different state. The second, in a meeting between nonbankers a nonbanker switches to a different

---

\(^6\) As emphasized by Randall Wright, "the reason nonbank notes will not be valued in any symmetric equilibrium is that no nonbank would ever produce to get a note issued by another nonbank because he can always issue his own note, which is by assumption a perfect substitute in a symmetric equilibrium."

\(^7\) Note holding is not a state variable for a banker because all banks are treated equally and a banker can always print a new one.
state if and only if the other does.

Note that this conditions must hold only when nonbankers meet. Only in those meetings are initial note holdings necessarily preserved.

From now on we will focus on a particular group of stationary and symmetric inside money allocation, namely simple allocations, described as follows: in all single-coincidence meetings between nonbankers, there is production in exchange of a note if and only if that is consistent with preservation of note holdings in the meeting and the unit upper bound on holdings. A nonbanker produces for a banker and receives a note if and only if the nonbanker does not have a note. A banker ever produces for a nonbanker - if the nonbanker has a note, the bank receives and destroys it - and for another bank. In all other meetings nothing happens.

2.1.3 Implementable allocations

An allocation of this group will be called implementable if it satisfies some participation constraints.

Before setting constraints, we will assume that each person has the option to behave autarkically in any meeting.

In order to express participation constraints in terms of discounted expected utilities, we first define \( v^k_{i<p>}, \) the expected discounted utility for a person with identity \( k \) who begins a period \( p \) in state \( i \).

The first constraints are restrictions on production by bankers,

\[
-y_{b<l>} + \beta \left\{ \pi v_{b<h>^{l'}} + (1 - \pi) v_{b<h_1>}^{l'} \right\} \geq 0 \tag{2}
\]
\[
-y_{b<h>} + \beta \left\{ \psi v_{b<h'}^{h'} + (1 - \psi) v_{b<l_1>}^{h'} \right\} \geq 0, \tag{3}
\]

for \( l \in \{l_1, \ldots, l_n\}, h \in \{h_1, \ldots, h_m\}, l' \) denoting the low subperiod after \( l \), \( h' \) denoting the high subperiod after \( h \), \((m,j) \in \{(b,\cdot), (n,0), (n,1)\} \).

The right-hand side of (2) and (3) are zero because the highest feasible punishment prevents banks from consuming after a defection. This is so because the identity of a bank defector becomes public information. Hence, new money issued by them can be declared worthless. Moreover, each person meeting with bank defectors can offer an arbitrarily small quantity of output for money they may have acquired from others, knowing they cannot find a better deal in the future if others hold up the punishment. It is then an
equilibrium to offer zero consumption to a bank defector forever since people have no incentives to deviate unilaterally from this punishment scheme.

We require that nonbankers have non-negative gains from trade when they consume. The next constraints summarize this:

\[ u_l(y_{1\left< l \right>}^k) + \beta \left\{ \pi v_{0< l'}^n + (1 - \pi)v_{1< h_l}^n \right\} \geq \beta \left\{ \pi v_{1< l'}^n + (1 - \pi)v_{1< h_l}^n \right\} \]  \hspace{1cm} (4)

\[ u_h(y_{1<h>1}^h) + \beta \left\{ \psi v_{0< l'}^n + (1 - \psi)v_{1< h_l}^n \right\} \geq \beta \left\{ \psi v_{1< l'}^n + (1 - \psi)v_{1< h_l}^n \right\} \]  \hspace{1cm} (5)

\[ u_l(y_{0< l'}^m) + \beta \left\{ \pi v_{1< l'}^n + (1 - \pi)v_{0< h_{l'}}^n \right\} \geq \beta \left\{ \pi v_{0< l'}^n + (1 - \pi)v_{0< h_{l'}}^n \right\} \]  \hspace{1cm} (6)

\[ u_h(y_{0<h>1}^m) + \beta \left\{ \psi v_{0< l'}^n + (1 - \psi)v_{0< h_{l'}}^n \right\} \geq \beta \left\{ \psi v_{0< l'}^n + (1 - \psi)v_{0< h_{l'}}^n \right\} \]  \hspace{1cm} (7)

for \( l \in \{l_1, ..., l_m\} \), \( h \in \{h_1, ..., h_m\} \), \( l' \) denoting the low subperiod after \( l \), \( h' \) denoting the high subperiod after \( h \), \( (k, i) \in \{(b, \cdot), (n, 0)\} \).

The next constraint requires that nonbankers have non-negative gains from trade when they produce,

\[ -y_{l< l'}^m + \beta \left\{ \pi v_{1< l'}^n + (1 - \pi)v_{1< h_l}^n \right\} \geq \beta \left\{ \pi v_{1< l'}^n + (1 - \pi)v_{1< h_l}^n \right\} \]  \hspace{1cm} (8)

\[ -y_{l<h>1}^m + \beta \left\{ \psi v_{1< l'}^n + (1 - \psi)v_{1< h_l}^n \right\} \geq \beta \left\{ \psi v_{1< l'}^n + (1 - \psi)v_{1< h_l}^n \right\} \]  \hspace{1cm} (9)

\[ -y_{0< l'}^m + \beta \left\{ \pi v_{1< l'}^n + (1 - \pi)v_{0< h_{l'}}^n \right\} \geq \beta \left\{ \pi v_{0< l'}^n + (1 - \pi)v_{0< h_{l'}}^n \right\} \]  \hspace{1cm} (10)

\[ -y_{0<h>1}^m + \beta \left\{ \psi v_{1< l'}^n + (1 - \psi)v_{1< h_l}^n \right\} \geq \beta \left\{ \psi v_{1< l'}^n + (1 - \psi)v_{1< h_l}^n \right\} \]  \hspace{1cm} (11)

for \( l \in \{l_1, ..., l_m\} \), \( h \in \{h_1, ..., h_m\} \), \( l' \) denoting the low subperiod after \( l \), \( h' \) denoting the high subperiod after \( h \), \( (m, j) \in \{(b, \cdot), (n, 1)\} \).

We will also assume that a nonbanker does not freely dispose of money. As a consequence and by restrictions \(8\) and \(9\) , nonbankers who possess money will never produce until she consumes and gives up money.

The stationarity across states implies that the inflow into state 1 must equal the outflow from state 1 for

---

8Constraints \(4\) and \(5\) are never binding in all parameter space, while constraints \(6\) and \(7\) implies that \(y_{0<l'}^m y_{0<h>1}^m \geq 0\).
them, i.e., for implementable simple allocations we have\(^9\):

\[
x^n_0 [x^n_0 + x^n_1] = x^n_1 [x^n_0 + x^n_0]
\]  
(12)

By equation (1),

\[
x^n_0 = \frac{(1 - B)}{2}
\]  
(13)

The stationarity also implies that we can write, for low period states:

\[
v^n_{0\langle l \rangle} = \frac{x^n_b}{S} u_l \left( y^n_{b0\langle l \rangle} \right) + \frac{x^n_b}{S} \left( -y^n_{bn\langle l \rangle} \right) + \frac{x^n_b}{S} \left( -y^n_{01\langle l \rangle} \right) + \\
+ \beta \left\{ \left( \frac{x^n_b + x^n_0}{S} \right) [\pi v^n_{1\langle l' \rangle} + (1 - \pi) v^n_{0\langle b1 \rangle}] \right\} \left( 1 - \frac{x^n_b + x^n_1}{S} \right) \]  
(14)

\[
v^n_{1\langle l \rangle} = \frac{x^n_b}{S} u_l \left( y^n_{b1\langle l \rangle} \right) + \frac{x^n_b}{S} u_l \left( y^n_{bn0\langle l \rangle} \right) + \\
+ \beta \left\{ \left( \frac{x^n_b + x^n_0}{S} \right) [\pi v^n_{0\langle l' \rangle} + (1 - \pi) v^n_{0\langle b1 \rangle}] \right\} \left( 1 - \frac{x^n_b + x^n_1}{S} \right) \]  
(15)

\[
v^b_{\langle l \rangle} = \frac{x^b_b}{S} u_l \left( y^b_{b0\langle l \rangle} \right) + \frac{x^b_0}{S} u_l \left( y^n_{bn0\langle l \rangle} \right) + \frac{x^b_b}{S} \left( -y^n_{bn\langle l \rangle} \right) + \frac{x^b_0}{S} \left( -y^n_{01\langle l \rangle} \right) + \\
+ \frac{x^n_1}{S} \left( -y^n_{1\langle b1 \rangle} \right) + \beta \left\{ \pi v^b_{\langle l' \rangle} + (1 - \pi) v^b_{\langle b1 \rangle} \right\} ,
\]  
(16)

for \( l \in \{ l_1, ..., l_{nl} \} \) and \( l' \) denoting the low subperiod after \( l \).

In the same way, for high period states:

\[
v^n_{0\langle h \rangle} = \frac{x^n_b}{S} u_h \left( y^n_{b0\langle h \rangle} \right) + \frac{x^n_b}{S} \left( -y^n_{bn\langle h \rangle} \right) + \frac{x^n_0}{S} \left( -y^n_{bn0\langle h \rangle} \right) + \\
+ \beta \left\{ \left( \frac{x^n_b + x^n_1}{S} \right) [\pi v^n_{1\langle h' \rangle} + (1 - \pi) v^n_{1\langle b1 \rangle}] \right\} \left( 1 - \frac{x^n_b + x^n_1}{S} \right) \]  
(17)

\(^9\)We have built into (12) that nonbankers receive money when they are producers and meet potential consumers and give up money when they are consumers and meet potential consumers.
\[ v^n_{1<h>} = \frac{x^b}{S} u_h(y^b_{1<h>}) + \frac{x^0}{S} u_h(y^0_{1<h>}) + \\
+ \beta \left[ \left( \frac{x^b + x^n}{S} \right) [\psi v^n_{0<h'} > + (1 - \psi) v^n_{0<t_1>}] + \left( 1 - \frac{x^b + x^0}{S} \right) [\psi v^n_{1<h'} > + (1 - \psi) v^n_{1<t_1>}] \right] \] (8)

\[ v^b_{<h>} = \frac{x^b}{S} u_h(y^b_{<h>}) + \frac{x^0}{S} u_h(y^0_{<h>}) + \frac{x^b}{S} (-y^b_{<h>}) + \\
+ \frac{x^n}{S} (-y^n_{<h>}) + \beta \left[ \psi v^b_{<h'} > + (1 - \psi) v^b_{<t_1>} \right] \] (19)

for \( h \in \{h_1, ..., h_{n_h}\} \) and \( h' \) denoting the low subperiod after \( h \).

Notice that for a given allocation, (14) up to (19) consist of \( 3(n_l + n_h) \) linear equations in all expected discounted utilities. Those equations have a unique solution. Among the ways to establish that is by a trivial contraction mapping argument.

Now if we define the \((5(n_l + n_h) \times 1)\) vector \( y \) as:

\[ y = \left[ \begin{array}{c} y_l \\ y_h \end{array} \right], \text{ for} \]

\[ y_l \quad (5n_l \times 1) = \left[ y^{bb}_{<l_1>} \ldots y^{bb}_{<l_{n_l}>} y^{bn}_{0<l_1>} \ldots y^{bn}_{0<l_{n_l}>} y^{bl}_{1<l_1>} \ldots y^{bl}_{1<l_{n_l}>} y^{bn}_{0<l_1>} \ldots y^{bn}_{0<l_{n_l}>} \right] \]

and

\[ y_h \quad (5n_h \times 1) = \left[ y^{bb}_{<h_1>} \ldots y^{bb}_{<h_{n_h}>} y^{bn}_{0<h_1>} \ldots y^{bn}_{0<h_{n_h}>} y^{bn}_{0<h_1>} \ldots y^{bn}_{0<h_{n_h}>} \right] \].

It follows that \( v \), the \((3(n_l + n_h) \times 1)\) vector which elements contain all the expected discounted utility, is explicitly expressed in terms of an allocation and transition variables by:

\[ v(y) = \left[ \begin{array}{c} v_l(y) \\ v_h(y) \end{array} \right] = M^{-1} Ar(y), \text{ where} \]

\[ v_l(y) \quad (3n_l \times 1) = \left[ v^m_{0<l_1>} \ldots v^m_{0<l_{n_l}>} v^m_{1<l_1>} \ldots v^m_{1<l_{n_l}>} v^b_{<l_1>} \ldots v^b_{<l_{n_l}>} \right] \]

and
\[ v_h(y) = \begin{bmatrix} v^n_{0<h_1>} & \cdots & v^n_{0<h_n>} & v^n_{1<h_1>} & \cdots & v^n_{1<h_n>} & v^b_{<h_1>} & \cdots & v^b_{<h_n>} \end{bmatrix}^T, \]

and, for the matrix
\[ M_{(3(n_l+n_h)x3(n_l+n_h))} = I_{3(n_l+n_h)} - T_1 - T_2 \]

where,
\[ I_{3(n_l+n_h)} \text{ is the } (3(n_l+n_h)x3(n_l+n_h)) \text{ identity matrix,} \]

\[
T_1_{(3(n_l+n_h)x3(n_l+n_h))} = \frac{\beta}{4} \begin{bmatrix}
-(x^b + x^n_1)\Pi_1 & (x^b + x^n_1)\Pi_1 & 0 & (x^b + x^n_1)\Pi_2 & (x^b + x^n_1)\Pi_2 & 0 \\
(x^b + x^n_0)\Pi_1 & -(x^b + x^n_0)\Pi_1 & 0 & (x^b + x^n_0)\Pi_2 & -(x^b + x^n_0)\Pi_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-(x^b + x^n_1)\Psi_1 & (x^b + x^n_1)\Psi_1 & 0 & -(x^b + x^n_1)\Psi_2 & (x^b + x^n_1)\Psi_2 & 0 \\
(x^b + x^n_0)\Psi_1 & -(x^b + x^n_0)\Psi_1 & 0 & (x^b + x^n_0)\Psi_2 & -(x^b + x^n_0)\Psi_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix},
\]

and
\[
T_2_{(3(n_l+n_h)x3(n_l+n_h))} = \beta \begin{bmatrix} 
I_3 \otimes \Pi_1 & I_3 \otimes \Pi_2 \\
I_3 \otimes \Psi_2 & I_3 \otimes \Psi_1 
\end{bmatrix},
\]

the matrix
\[
A_{(3(n_l+n_h)x10(n_l+n_h))} = \frac{1}{3} \begin{bmatrix} 
A_l & 0 \\
0 & A_h 
\end{bmatrix},
\]

where,
\[
A_l_{(3n_lx10n_l)} = \begin{bmatrix} 
0 & x^b_1I_{u_l} & 0 & 0 & 0 & 0 & 0 & 0 & x^b_1I_{n_l} & x^n_1I_{n_l} \\
0 & 0 & x^b_1I_{u_l} & 0 & x^n_0I_{n_l} & 0 & 0 & 0 & 0 & 0 \\
x^b_1I_{n_l} & 0 & 0 & x^n_0I_{n_l} & 0 & x^b_1I_{u_l} & x^n_0I_{u_l} & x^n_0I_{n_l} & 0 & 0 \\
0 & x^b_1I_{n_h} & 0 & 0 & 0 & 0 & 0 & 0 & x^b_1I_{n_h} & x^n_1I_{n_h} \\
0 & 0 & x^b_1I_{u_h} & 0 & x^n_0I_{n_h} & 0 & 0 & 0 & 0 & 0 \\
x^b_1I_{n_h} & 0 & 0 & x^n_0I_{n_h} & 0 & x^b_1I_{n_h} & x^n_0I_{n_h} & x^n_1I_{n_h} & 0 & 0 
\end{bmatrix}
\]

and the \((10(n_l+n_h)x1)\) vector of pay-offs
\[ r(y) = \begin{bmatrix} r_l(y) \\
r_h(y) \end{bmatrix}, \]

where,
\[ r_l(y) = \begin{bmatrix} u_l(y_l) \\
-y_l \end{bmatrix} \text{ and } r_h(y) = \begin{bmatrix} u_h(y_h) \\
-y_h \end{bmatrix} \]

where \(u_l\) and \(u_h\) operates, respectively, \(y_l\) and \(y_h\) element-wise.
2.1.4 The maximization problem

The optimization problem is to maximize a welfare function over the set of implementable, simple, stationary and symmetric inside money allocations. The representative-agent criterion is

\[ w = \sum_{k, i, p} \varphi_{<p>} x_{k}^{i} v_{p}^{k} \]  \hspace{1cm} (22)

where \( \varphi \) is the invariant distribution of \( P \), \( (k, i) \in \{(b, \cdot), (n, 0), (n, 1)\} \) and \( p \in \{l_1, ..., l_n, h_1, ..., h_n\} \).

We can rewrite equation (22) as

\[ w = \vartheta' v(y) \]  \hspace{1cm} (23)

where the \((1 \times 3(n_l + n_h))\) vector \( \vartheta' \) is:

\[ \vartheta' = \left[ x \otimes \varphi_l \quad x \otimes \varphi_h \right] \]

for the vector of the distribution of bankers and nonbankers across states \( x \) = \([x_0^n \quad x_1^n \quad x_b]\),

the vector \( \varphi_l \) = \([\varphi_{<l_1>} \quad \varphi_{<l_2>} \quad \cdots \quad \varphi_{<l_{n_l}>}] \)

and

the vector \( \varphi_h \) = \([\varphi_{<h_1>} \quad \varphi_{<h_2>} \quad \cdots \quad \varphi_{<h_{n_h}>}] \), which were taken from the invariant distribution \( \varphi \) of \( P \).

Another way to write the welfare criterion is, by equations (13) and (14)-(19),

\[ w = \sum_p \left\{ \left[ \sum_{k, m, i, j} x_{k}^{i} x_{m}^{j} \frac{z(y_{ij}^{km})}{S} \right] \chi_p \right\} \]  \hspace{1cm} (24)

where \( (k, m, i, j) \in \{(b, b, \cdot, \cdot), (b, n, \cdot, \cdot), (n, b, \cdot, \cdot), (n, n, 0, \cdot), (n, n, 1, \cdot)\} \), \( p \in \{l_1, ..., l_n, h_1, ..., h_n\} \),

\( z(y_{ij}^{km}) = u(y_{ij}^{km}) - y_{ij}^{km} \), \( \chi_p \) is a nonnegative parameter that depends on \( \beta, \pi \) and \( \psi \), and is the present value of a flow of an unit of utility whenever period state is \( p \) and zero everywhere else, over a horizon that goes to infinite.\(^{10}\)

The core of equation (24) specifies the probability of a single-coincidence meeting and the pay-off derived

\(^{10}\)Note that \( \sum_p \chi_p = \frac{1}{1-\beta} \).
from that meeting to the economy. Hence, \( w \) can be understood as the present value expected in this economy.

To clarify a little what was said, if we set \( n_l = n_h = 1 \), then we would have

\[
\chi_l = \frac{(1 - \psi) [1 - \beta \psi] + \beta (1 - \pi)}{(2 - \pi - \psi) [(1 - \beta \pi) (1 - \beta \psi) - \beta^2 (1 - \pi)(1 - \psi)]}
\]

and

\[
\chi_h = \frac{(1 - \pi) [(1 - \beta \pi) + \beta (1 - \psi)]}{(2 - \pi - \psi) [(1 - \beta \pi) (1 - \beta \psi) - \beta^2 (1 - \pi)(1 - \psi)]}.
\]

If the period state were mostly high from now on, i.e., \( \pi \) low and \( \psi = 1 \), then \( \chi_l = 0 \) and \( \chi_h = \frac{1}{1 - \beta} \). This means that economic policy should focus only on high period states.

On the other hand, if the period state were mostly low from now on, i.e., \( \psi \) low and \( \pi = 1 \), then \( \chi_l = \frac{1}{1 - \beta} \) and \( \chi_h = 0 \). This means that economic policy should focus only on low period states.

**Proposition 1** The optimization problem over the set of implementable, simple, stationary and symmetric inside money allocations has a unique solution that is continuous in \( \theta = (B, \pi, \psi, \beta) \in \Theta = \{[0, 1]^3 \times [0, 1]\} \).

**Proof.** Appendix A. ■

From (24), it is easy to see that \( y^* \), the first-best level of output is \( y_l^* = \frac{\log(A_l)}{A_l} \), when period state is low and \( y_h^* = \frac{\log(A_h)}{A_h} \), when period state is high.
3 Real Shocks

In this chapter, we consider the Cavalcanti-Forno’s model for seasonality due to real shocks on preferences with no persistence, $\pi = \psi = 0$, and no monetary policy improvement, $n_l = n_h = 1$. CW is a particular case where there is no real shocks, i.e., $\gamma_l = \gamma_h = \gamma$.

Here, we study values of $\beta$ ranging from 0.4 to 0.9 in order to map the parameter space and we solve the optimization problem in a fashion that good insights into optimum and into the role played by banks are easy to access. Moreover, this procedure supplies good test cases for next chapter, which allows both for persistence and for monetary policy role.

3.1 CW

In this section we revisit CW in order to access some of its main insights, namely the optimality of paying interest on inside money in a banking model. Moreover, we believe this will be useful for the understanding of CF in the next sections.

Remembering that in CW case we have $n_l = n_h = 1$ and $\gamma_l = \gamma_h = \gamma$, so we can suppress all period state representation. The first step will be to rewrite the welfare function, all important participation constraints and value functions.

In this particular case, from (24)-(26), the welfare function $w$ is such that:

$$S(1-\beta)w = x^b x^b [u(y^{bb}) - y^{bb}] + x^b x^n [u(y^{nb}_0) + u(y^{bn}_0) - y^{bn}_0] +$$

$$+ x^n x^n [u(y^{nn}_0) - y^{nn}_0] + x^n x^n [u(y^{nn}_1) - y^{nn}_0].$$

The relevant participation constraints are, from (2)-(11):

$$g_1 = -y^{bb} + \beta v^b \geq 0$$
$$g_2 = -y^{bn}_0 + \beta v^b \geq 0$$
$$g_3 = -y^{bn}_1 + \beta v^b \geq 0$$
$$g_4 = -y^{nb}_0 + \beta (v^n_0 - v^n_0) \geq 0$$
$$g_5 = -y^{nb}_1 + \beta (v^n_1 - v^n_0) \geq 0$$
$$g_6 = y^{bn}_0 \geq 0$$
From (14) - (19), value functions must satisfy:

\[ S(1 - \beta)v^b = x^b \left[ u \left( y^{b_b} \right) - y^{b_b} \right] + x^n_0 \left[ u \left( y^{b_n} \right) - y^{b_n}_0 \right] + x^n_1 \left[-y^{b_n}_1 \right] \]

\[ \left[ S(1 - \beta) + \beta \left( 1 + x^b \right) \right] \left( v^n_1 - v^n_0 \right) = x^b \left[ u \left( y^{b_1} \right) - u \left( y^{b_0} \right) + y^{b_n}_0 \right] + x^n_0 \left[ u \left( y^{b_0} \right) \right] + x^n_1 \left[ y^{b_1}_0 \right] \]

This two equations supply

\[ \frac{\partial v^b}{\partial y^{b_b}} = \frac{x^b \left[ u \left( y^{b_b} \right) - 1 \right]}{S(1 - \beta)} \]

\[ \frac{\partial v^b}{\partial y^{b_0}} = \frac{x^n_0}{S(1 - \beta)} \]

\[ \frac{\partial v^b}{\partial y^{b_1}} = \frac{x^n_1}{S(1 - \beta)} \]

\[ \frac{\partial v^b}{\partial y^{b_0}} = \frac{x^n_0 u \left( y^{b_0}_n \right)}{S(1 - \beta)} \]

\[ \frac{\partial v^b}{\partial y^{b_1}} = 0 \]

\[ \frac{\partial (v^n_1 - v^n_0)}{\partial y^{b_b}} = \frac{x^b \left[ u \left( y^{b_1}_n \right) \right]}{S(1 - \beta) + \beta \left( 1 + x^b \right)} \]

\[ \frac{\partial (v^n_1 - v^n_0)}{\partial y^{b_0}} = \frac{x^b u \left( y^{b_0}_n \right)}{S(1 - \beta) \left( 1 + x^b \right)} \]

\[ \frac{\partial (v^n_1 - v^n_0)}{\partial y^{b_1}} = \frac{x^n_1}{S(1 - \beta) + \beta \left( 1 + x^b \right)} \]

\[ \frac{\partial (v^n_1 - v^n_0)}{\partial y^{b_{n1}}} = \frac{x^n_0 u \left( y^{b_{n1}}_0 + x^n_1 \right)}{S(1 - \beta) + \beta \left( 1 + x^b \right)} \]

Figure (1) is a mapping for \( \gamma = 3.0 \) and \( S = 3 \), using a grid of 5,782 points over \( \beta \in [0.40, 0.98] \) and \( B \in [0.01, 0.98] \) and figure (2) is a mapping for \( \gamma = 2.5 \) and \( S = 3 \), using a grid of 4,998 points over \( \beta \in [0.48, 0.98] \) and \( B \in [0.01, 0.98] \).
Figure 1 - CW mapping for $\gamma = 3.0$ and $S = 3$, using a grid of 5,782 points over $\beta \in [0.40, 0.98]$ and $B \in [0.01, 0.98]$.

Figure 2 - CW mapping for $\gamma = 2.5$ and $S = 3$, using a grid of 4,998 points over $\beta \in [0.48, 0.98]$ and $B \in [0.01, 0.98]$.

In both figures, we can identify and draft - as done in figure (3) - eight regions, namely I,A,B,C,D,E,R,1.
Tables (1) and (2) display the computed values of output relative to $g^*$, in percentage points, for the previous cases.

<table>
<thead>
<tr>
<th>Regions</th>
<th>I</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>R</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>0.50</td>
<td>0.02</td>
<td>0.02</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.02</td>
<td>0.10</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.79</td>
<td>0.57</td>
<td>0.54</td>
<td>0.60</td>
<td>0.48</td>
<td>0.45</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>$y^{bb}$</td>
<td>100.0000</td>
<td>99.4516</td>
<td>91.3943</td>
<td>100.0000</td>
<td>100.0000</td>
<td>99.2092</td>
<td>78.9251</td>
<td>83.0263</td>
</tr>
<tr>
<td>$y^{bn}_0$</td>
<td>100.0000</td>
<td>99.4516</td>
<td>91.3943</td>
<td>65.2353</td>
<td>43.9983</td>
<td>37.8337</td>
<td>78.9251</td>
<td>83.0263</td>
</tr>
<tr>
<td>$y^{bn}_1$</td>
<td>100.0000</td>
<td>99.4516</td>
<td>91.3943</td>
<td>116.6436</td>
<td>111.4107</td>
<td>99.2092</td>
<td>78.9251</td>
<td>83.0263</td>
</tr>
<tr>
<td>$y^{nb}_0$</td>
<td>100.0000</td>
<td>100.1701</td>
<td>101.4593</td>
<td>65.9738</td>
<td>52.6100</td>
<td>49.9180</td>
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<td>76.2535</td>
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<tr>
<td>$y^{nn}_{01}$</td>
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<td>100.0000</td>
<td>100.0139</td>
<td>65.9738</td>
<td>52.6100</td>
<td>49.9180</td>
<td>86.4856</td>
<td>76.2535</td>
</tr>
</tbody>
</table>

Table 1 - Percent of relative output for selected values of $B$ and $\beta$ with $\gamma = 3.0$ and $S = 3$ in CW.
<table>
<thead>
<tr>
<th>Regions</th>
<th>I</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>R</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>0.50</td>
<td>0.02</td>
<td>0.02</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.92</td>
<td>0.69</td>
<td>0.65</td>
<td>0.79</td>
<td>0.60</td>
<td>0.50</td>
<td>0.60</td>
<td>0.48</td>
</tr>
<tr>
<td>$y_{bb}^0$</td>
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<td>87.5756</td>
<td>100.0000</td>
<td>100.0000</td>
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<td>72.3817</td>
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</tr>
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<td>$y_{b0}^{ln}$</td>
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<td>87.5756</td>
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<td>45.2889</td>
<td>22.5670</td>
<td>72.3817</td>
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<tr>
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<td>87.5756</td>
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<td>113.2598</td>
<td>77.5280</td>
<td>72.3817</td>
<td>44.2102</td>
</tr>
<tr>
<td>$y_{00}^{nb}$</td>
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<td>102.5543</td>
<td>80.7831</td>
<td>56.9754</td>
<td>51.0179</td>
<td>83.2648</td>
<td>44.2102</td>
</tr>
<tr>
<td>$y_{00}^{nb}$</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0429</td>
<td>80.7831</td>
<td>56.9754</td>
<td>51.0179</td>
<td>83.2648</td>
<td>44.2102</td>
</tr>
</tbody>
</table>

Table 2 - Percent of relative output for selected values of $B$ and $\beta$ with $\gamma = 2.5$ and $S = 3$ in CW.

We define interest rates as $r = \frac{y_{bb}}{y_{0b}} - 1$, as Cavalcanti and Wallace (2001). Figure (4) displays a mesh for $r$ for $\gamma = 3.0$ in CW.

![Figure 4 - CW mesh for $r$ (%) for $\gamma = 3.0$ and $S = 3$, using a grid of 5,782 points over $\beta \in [0.40, 0.98]$ and $B \in [0.01, 0.98]$.](image-url)

In region I all constraints slack, we get a unrestricted maximum with first-best level of output $y^*$ and, consequently, $r = 0$. If we lower the value of $\beta$, we will reach region A for lowers values of $B$ or region C elsewhere.
In region C nonbank production constraints \((g_4 \text{ and } g_5)\) bind first and individual will produce less than \(y^*\). In order to lose that constraint, banks increase return on money by decreasing the amount produced to nonbanks without money, i.e. gifts, and increasing the amount produced to nonbanks holding money\(^{11}\). As a result, \(r > 0\) and \(y^{nb}_0 = y^{nn}_0\). That is, it is optimal to use all of nonbank production possibilities in both meetings.

The same pattern can be found in regions D and E, if we keep reducing \(\beta\). The only difference is that in region D bank production constraint \(g_3\) binds and banks have no incentives to fully increase interest on money, and in region E, additionally, bank production constraint \(g_1\) binds and banks have no incentives to produce \(y^*\) for another bank.

In region A, i.e., for low \(B\), bank production constraints \((g_1, g_2 \text{ and } g_3)\) bind first and banks will produce less than \(y^*\). In order to lose that constraint and slow down bank production fall, nonbanks are asked to raise expected utility of banks by increasing the amount produced to them\(^{12}\). As a result, \(r < 0\) and \(y^{nb}_0 \neq y^{nn}_0\).

The same pattern can be found in region B, if we keep reducing \(\beta\). The only difference is that in region B nonbank production constraint \(g_4\) binds, nonbanks have no incentives to fully increase the amount produced to banks as wished by the last, and nonbank raises production to another nonbank holding money\(^{13}\).

In region R, all production constraints bind. In region 1, bank gifts are no longer restricted but nonbank without money consumption constraint, i.e., \(g_6\) turns out to be binding. For that reason Cavalcanti and Wallace (2001) called it “no-gifts” region.

### 3.2 CF - first visit

In this section, we consider the Cavalcanti-Forno’s model for seasonality due to real shocks on preferences with no persistence, \(\pi = \psi = 0\), and no monetary policy improvement, \(n_l = n_h = 1\). There are real shocks on preferences, i.e., \(\gamma_l > \gamma_h\).

\(^{11}\)Note that \(\frac{\partial (v^n_1 - v^n_0)}{\partial y^{bb}}|_{y^*} = 0\), \(\frac{\partial (v^n_1 - v^n_0)}{\partial y^{bn}}|_{y^*} < 0, \frac{\partial (v^n_1 - v^n_0)}{\partial y^{nn}}|_{y^*} > 0\) for a backward looking bank action.

\(^{12}\)Note that \(\frac{\partial p^b}{\partial y^{bn}}|_{y^*} = 0\) and \(\frac{\partial p^b}{\partial y^{nn}}|_{y^*} > 0\).

\(^{13}\)Note that \(\frac{\partial (v^n_1 - v^n_0)}{\partial y^{nn}}|_{y^*} > 0\).
In this case, from (24)-(26), the welfare function \( w \) is such that:

\[
S(1 - \beta)w = \frac{1}{2} \sum_{p \in \{l, h\}} \{ x^b \cdot x^h \left[ u_p \left( y_{-1}^b < p \right) - y_{-1}^b < p \right] + \\
+ x^b \cdot x^h_0 \left[ u_p \left( y_{-1}^b_0 < p \right) + u_p \left( y_{-1}^h_0 < p \right) - y_{-1}^b_0 < p \right] + \\
+ x^b \cdot x^h_1 \left[ u_p \left( y_{-1}^b_1 < p \right) - y_{-1}^b_1 < p \right] + x^h_0 \left[ u_p \left( y_{-1}^h_0 < p \right) - y_{-1}^h_0 < p \right] \}.
\]

The relevant participation constraints are, from (2)-(11):

\[
\begin{align*}
g_1 &= -y_{-1}^b < l > + \beta v^b_{< l >} \geq 0 \\
g_2 &= -y_{-1}^h < l > + \beta v^b_{< l >} \geq 0 \\
g_3 &= -y_{-1}^h < l > + \beta v^b_{< l >} \geq 0 \\
g_4 &= -y_{-1}^h < l > + \beta (v^n_{1 < l >} - v^n_{0 < l >}) \geq 0 \\
g_5 &= -y_{01}^n < l > + \beta (v^n_{1 < l >} - v^n_{0 < l >}) \geq 0 \\
g_6 &= y_{01}^n < l > \geq 0 \\
g_7 &= -y^b_{< l >} + \beta v^b_{< l >} \geq 0 \\
g_8 &= -y^h_{< l >} + \beta v^b_{< l >} \geq 0 \\
g_9 &= -y^h_{< l >} + \beta v^b_{< l >} \geq 0 \\
g_{10} &= -y^b_{< l >} + \beta (v^n_{1 < l >} - v^n_{0 < l >}) \geq 0 \\
g_{11} &= -y_{01}^n < l > + \beta (v^n_{1 < l >} - v^n_{0 < l >}) \geq 0 \\
g_{12} &= y_{01}^n < l > \geq 0
\end{align*}
\]

From (14) - (19), value functions must satisfy:

\[
S \left( 1 - \beta^2 \right) x^b_{< l >} = x^b \left[ u_p \left( y_{-1}^b < p \right) - y_{-1}^b < p \right] + x^h \left[ u_p \left( y_{-1}^b < p \right) - y_{-1}^b < p \right] + x^h_0 \left[ u_p \left( y_{0}^b < p \right) - y_{-1}^b < p \right] + x^h_1 \left[ -y_{-1}^h < l > \right] + \\
+ \beta \left\{ x^b \left[ u_p' \left( y_{-1}^b < p' \right) - y_{-1}^b < p' \right] + x^h \left[ u_p' \left( y_{0}^h < p' \right) - y_{0}^h < p' \right] + x^h_0 \left[ u_p' \left( y_{01}^h < p' \right) - y_{-1}^h < p' \right] + x^h_1 \left[ -y_{1}^h < p' \right] \right\}
\]

\[
S \left( 1 - \lambda^2 \beta^2 \right) \left( v^n_{1 < l >} - v^n_{0 < l >} \right) = x^b \left[ u_p \left( y_{01}^h < p \right) - u_p \left( y_{0}^h < p \right) + y^b_{0 < p} \right] + x^h \left[ u_p \left( y_{01}^h < p \right) + y^b_{0 < p} \right] + x^h_0 \left[ u_p \left( y_{01}^h < p \right) \right] + x^h_1 \left[ y_{01}^h < p \right] + \\
\lambda \beta \left\{ x^b \left[ u_p' \left( y_{01}^h < p' \right) - u_p' \left( y_{0}^h < p' \right) + y^b_{0 < p'} \right] + x^h \left[ u_p' \left( y_{01}^h < p' \right) + y^b_{0 < p'} \right] + x^h_0 \left[ u_p' \left( y_{01}^h < p' \right) \right] + x^h_1 \left[ y_{01}^h < p' \right] \right\}
\]

where \( \lambda = 1 - \frac{1 + x^b}{S} \), for \( p, p' \in \{l, h\} \) and \( p \neq p' \).

The first pair of equations supply for banks
\[ \frac{\partial b}{\partial n_{b,p}} = \frac{x^{n} u^{b}_{n_{b,p}}(y^{b}_{n_{b,p}}) - 1}{S(1 - \beta^{2})} \]
\[ \frac{\partial b}{\partial y^{n}_{n_{b,p}}} = - \frac{x^{n}}{S(1 - \beta^{2})} \]
\[ \frac{\partial b}{\partial y^{h}_{1,p}} = - \frac{x^{n}}{S(1 - \beta^{2})} \]
\[ \frac{\partial b}{\partial y^{h}_{0,p}} = \frac{x^{n} u^{b}_{n_{b,p}}(y^{h}_{0,p})}{S(1 - \beta^{2})} \]
\[ \frac{\partial b}{\partial y^{b}_{0,p}} = 0 \]
\[ \frac{\partial b}{\partial y^{b}_{1,p}} = \beta \frac{x^{n} u^{b}_{n_{b,p}}(y^{b}_{1,p}) - 1}{S(1 - \beta^{2})} \]
\[ \frac{\partial b}{\partial y^{h}_{0,p}} = - \beta \frac{x^{n}}{S(1 - \beta^{2})} \]
\[ \frac{\partial b}{\partial y^{h}_{1,p}} = - \beta \frac{x^{n}}{S(1 - \beta^{2})} \]
\[ \frac{\partial b}{\partial y^{n+1}_{0,p}} = \beta \frac{x^{n} u^{b}_{n+1_{n_{b,p}}} - 1}{S(1 - \beta^{2})} \]
\[ \frac{\partial b}{\partial y^{n+1}_{1,p}} = 0 \]

The last pair of equations supply for nonbanks

\[ \frac{\partial(v_{n+1}^{c,p} - v_{n}^{c,p})}{\partial y^{n}_{c,p}} = 0 \]
\[ \frac{\partial(v_{n+1}^{c,p} - v_{n}^{c,p})}{\partial y^{h}_{n_{c,p}}} = x^{n} u^{c}_{n_{c,p}}(y^{h}_{n_{c,p}}) \frac{S(1 - \beta^{2})}{S(1 - \lambda^{2} \beta^{2})} \]
\[ \frac{\partial(v_{n+1}^{c,p} - v_{n}^{c,p})}{\partial y^{h}_{1,c,p}} = x^{n} u^{c}_{n_{c,p}}(y^{h}_{1,c,p}) \frac{S(1 - \beta^{2})}{S(1 - \lambda^{2} \beta^{2})} \]
\[ \frac{\partial(v_{n+1}^{c,p} - v_{n}^{c,p})}{\partial y^{h}_{0,c,p}} = x^{n} \frac{S(1 - \beta^{2})}{S(1 - \lambda^{2} \beta^{2})} \]
\[ \frac{\partial(v_{n+1}^{c,p} - v_{n}^{c,p})}{\partial y^{h}_{0,c,p}} = x^{n} \frac{S(1 - \beta^{2})}{S(1 - \lambda^{2} \beta^{2})} \]

Figure (5) is a mapping for \( \gamma_{l} = 3.0, \gamma_{h} = 2.5 \) and \( S = 3 \), using a grid of 4,661 points over \( \beta \in [0.40, 0.98] \) and \( B \in [0.20, 0.98] \).
Figure 5 - CF mapping for $\gamma_l = 3.0$, $\gamma_h = 2.5$ and $S = 3$, using a grid of 4,661 points over $\beta \in [0.40, 0.98]$ and $B \in [0.20, 0.98]$.

In figure (6) we show a draft over the whole parameter space. The detached area in this figure represents the grid of the previous figure.

Figure 6 - CF mapping draft.

Table (3) displays the computed values of output relative to $y^*_l$ and $y^*_h$, in percentage points, for selected points in this case.
Table 3 - Percent of relative output for selected values of $B$ and $\beta$ with $\gamma_l = 3.0, \gamma_h = 2.5$ and $S = 3$ in CF.

We now define interest rates in period states low and high as, respectively, $r_{<l>} \equiv \frac{y_{0,b}^{\alpha}}{y_{0,b}^{\alpha} - 1}$ and $r_{<h>} \equiv \frac{y_{0,b}^{\alpha}}{y_{0,b}^{\alpha} - 1}$. Figures (7) and (8) display a mesh for $r_{<l>}$ and $r_{<h>}$, respectively, for this same case in CF, using a grid of 5,782 points over $\beta \in [0.40,0.98]$ and $B \in [0.01,0.98]$. 

<table>
<thead>
<tr>
<th>Regions</th>
<th>$I$</th>
<th>$C_h$</th>
<th>$C_{l2}$</th>
<th>$C_{l2}$</th>
<th>$D_h$</th>
<th>$E_h$</th>
<th>$D_l$</th>
<th>$E_l$</th>
</tr>
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<tbody>
<tr>
<td>$B$</td>
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<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.95</td>
<td>0.87</td>
<td>0.82</td>
<td>0.70</td>
<td>0.58</td>
<td>0.53</td>
<td>0.50</td>
<td>0.45</td>
</tr>
<tr>
<td>$y_{0,b}^{\alpha}_{&lt;l&gt;}$</td>
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<td>100.0000</td>
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</tr>
<tr>
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<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
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<tr>
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<td>42.0257</td>
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</tr>
<tr>
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<td>115.5372</td>
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<tr>
<td>$y_{0,b}^{\alpha}_{&lt;h&gt;}$</td>
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<td>63.1603</td>
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<td>60.9185</td>
<td>58.2567</td>
<td>54.9445</td>
</tr>
<tr>
<td>$y_{01,b}^{\alpha}_{&lt;h&gt;}$</td>
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<td>90.5658</td>
<td>82.1723</td>
<td>63.1603</td>
<td>50.4756</td>
<td>46.6139</td>
<td>44.6148</td>
<td>41.6981</td>
</tr>
</tbody>
</table>
From now on, we will focus on what happens in the region of the parameter space where nonbank production constraints bind first.

In region I all constraints slack, we get a unrestricted maximum with first-best level of output $y^*_l$ and $y^*_h$ in period states low and high, respectively. Consequently, $r_{<l>} = \frac{y^*_l}{y^*_h} - 1$ and $r_{<h>} = \frac{y^*_h}{y^*_l} - 1$.

If we lower the value of $\beta$, we will reach region $C_h$ where the production constraints of nonbanks in
period state high \((g_{10} \text{ and } g_{11})\) bind first. In order to loose that restriction and improve welfare, banks increase returns on money in both period states\(^14\) and nonbanks produce more than \(y^*_n\) in low states\(^15\). As a consequence, \(r_{<h>}\) will certainly increase but \(r_{<b>}\) not necessarily.

The same pattern will be found in regions \(C_{1f}\) and \(C_{12}\), if we keep reducing \(\beta\). In \(C_{1f}\) constraint \(g_5\) binds and in \(C_{12}\) constraints \(g_4\) and \(g_5\) bind. In this last case, it is optimal to use all of nonbank production possibilities in both meetings.

In regions \(D_h, E_h, D_l, E_l\) bank production constraints bind in the following cumulative order: \(g_9, g_7, g_3\) and \(g_1\).

The next step will be focus on a particular point in the grid and see what happens when we introduce subperiods and persistence in our analisys.

\(^{14}\) Note that \(\frac{\partial (v_{t<1>}-v_{t<1>})}{\partial y_{<h>}}\bigg|_{y=y^*} = 0\), \(\frac{\partial (v_{t<1>}-v_{t<1>})}{\partial y_{<1>}}\bigg|_{y=y^*} > 0\) for a backward looking bank action.

\(^{15}\) Note that \(\frac{\partial (v_{t<1>}-v_{t<1>})}{\partial y_{<1>}}\bigg|_{y=y^*} > 0\) for a forward looking nonbank action.
4 Persistence and Monetary Policy

In this chapter, we consider the Cavalcanti-Forno’s model for seasonality due to real shocks on preferences with persistence and monetary policy improvement.

In the next section, we will focus on the process of designing the algorithm used to solve the maximization problem. In the other sections, we will display optimum over a grid of $\pi, \psi \in \{0.01, 0.25, 0.50, 0.75, 0.99\}$ on a particular point in the parameter space.

4.1 The Algorithm

We seek to maximize the lagrangean, modified by exponential penalty functions,

$$L(y, \mu) = \vartheta' v(y) - \frac{1}{e} \mu' J(g(y))$$  \hspace{1cm} (27)

where the $(5(n_l + n_h)x1)$ vector $y$ is given by equation (20), the $(3(n_l + n_h)x1)$ vector of expected discounted utility can be expressed explicitly in terms of an allocation and transition variables by $v(y) = M^{-1} Ar(y)$, where the $(1x6(n_l + n_h))$ vector of lagrange multipliers is $\mu'$, $j(g(y))$ is the composite function of exponential penalty functions:

$$j(g(y)) = \begin{bmatrix} e^{-c_{g1}(y)} - 1 \\ e^{-c_{g2}(y)} - 1 \\ \vdots \\ e^{-c_{g6(n_l+n_h)}(y)} - 1 \end{bmatrix}$$

for the elements of $(6(n_l + n_h)x1)$ vector $g(y) = e(y) + \beta Fv(y)$, which denotes the “slack” of the constraints derived from (2),(3),(6),(7),(10),(11) where

$$e(y) = \begin{bmatrix} e_l(y) \\ e_h(y) \end{bmatrix}$$ for the vectors
the $(6(n_l + n_h) \times 3(n_l + n_h))$ matrix $F$ is
\[
F = \begin{bmatrix}
F_l \\
F_h
\end{bmatrix}
\]
for:
\[
F_l = \begin{bmatrix}
0 & 0 & \Pi_1 & 0 & 0 & \Pi_2 \\
0 & 0 & \Pi_1 & 0 & 0 & \Pi_2 \\
0 & 0 & \Pi_1 & 0 & 0 & \Pi_2 \\
-\Pi_1 & \Pi_1 & 0 & -\Pi_2 & \Pi_2 & 0 \\
-\Pi_1 & \Pi_1 & 0 & -\Pi_2 & \Pi_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
and
\[
F_h = \begin{bmatrix}
0 & 0 & \Psi_2 & 0 & 0 & \Psi_1 \\
0 & 0 & \Psi_2 & 0 & 0 & \Psi_1 \\
0 & 0 & \Psi_2 & 0 & 0 & \Psi_1 \\
-\Psi_2 & \Psi_2 & 0 & -\Psi_1 & \Psi_1 & 0 \\
-\Psi_2 & \Psi_2 & 0 & -\Psi_1 & \Psi_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
At last, $c > 0$ is a parameter used for numerical stability.

Bertsekas\textsuperscript{16} discusses in detail the convergence properties of penalty functions and recommends a simple scheme for updating $\mu$ - so, we chose exponential updating as above:

\[
\mu_j^{(m+1)} = \mu_j^{(m)} e^{-\gamma_j^{(m)}}, \quad 1 \leq j \leq 6(n_l + n_h)
\]
where superscript $(m)$ means $m$-th iteration.

\textsuperscript{16}Bertsekas
Provided that good initial guesses for $\mu$ and $y$ can be found, and we choose the parameters $c$ and $\epsilon$ in order to improve the stability of convergence, a Newton-based search scheme works quite well.

Substituting (21) on (27), we get:

$$L(y, \mu) = \theta'M^{-1}Ar(y) - \frac{1}{c} \mu'j(g(y))$$

Differentiating with respect to $y'$, produces:

$$D[L(y, \mu)],_{(1 \times 5(n_l+n_h))} = \theta'M^{-1}A D[r(y)] - \frac{1}{c} \mu' D[j(g(y))]$$

(28)

where, by the chain rule:

$$D[j(g(y))] = D[j(g)] D[g(y)]$$

for:

$$D[j(g)] = \begin{bmatrix} e^{-c g_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-c g_6(n_l+n_h)} \end{bmatrix}$$

and

$$D[g(y)] = D[g(y)] + \beta FM^{-1}A D[r(y)]$$

Vectorizing (28) and differentiating again with respect to $y'$, produces the Hessian of $L$:

$$H[L(y, \mu)],_{(5(n_l+n_h) \times 5(n_l+n_h))} = (I_{5(n_l+n_h)} \otimes \theta'M^{-1}A) H[r(y)] - \frac{1}{c} (I_{5(n_l+n_h)} \otimes \mu') H[j(g(y))]$$

(29)

where, by the chain rule:

$$H[j(g(y))] = (I_{6(n_l+n_h)^2} \otimes D[g(y)]) H[j(g)] D[g(y)] +$$

$$+ (D[j(g)] \otimes I_{5(n_l+n_h)}) H[g(y)]$$

for:

$$H[j(g)],_{(36(n_l+n_h)^2 \times 6(n_l+n_h))} = c^2 \begin{bmatrix} e^{-c g_1 \epsilon_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-c g_6(n_l+n_h) \epsilon_6(n_l+n_h)} \end{bmatrix},$$

where $\epsilon_j$ is the column vector of size

$\pi, \psi, n_l, n_h, S, \beta$ and $B$. Besides, it’s quite crucial to initiate doing $\epsilon = c$ and raise $\epsilon$ as $g^{(m)}$ gets closed to optimum, i.e., when Newton’s step goes to zero.

---

17 The number of iterations across this optimization scheme depends mostly on the values of $\pi, \psi, n_l, n_h, S, \beta$ and $B$. Besides, it’s quite crucial to initiate doing $\epsilon = c$ and raise $\epsilon$ as $g^{(m)}$ gets closed to optimum, i.e., when Newton’s step goes to zero.

18 See Magnus and Neudecker, 1988, 91.

19 See Magnus and Neudecker, 1988, 110.
\(6(n_l + n_h)\), which has 1 as the \(j\)th element and zero everywhere else, and\(^{20}\)

\[
\mathbf{H}[g(y)] = \beta(I_{5(n_l+n_h)} \otimes F M^{-1} A) \mathbf{H}[r(y)].
\]

The \(m\)th step in the iteration updates the estimate of \(y_{opt}\), by solving the system:

\[
\mathbf{H}\left[\mathbf{L}(y^{(m)}, \mu^{(m)})\right]\left\{y^{(m+1)} - y^{(m)}\right\} = \mathbf{D}\left[\mathbf{L}(y^{(m)}, \mu^{(m)})\right]' \quad (29)
\]

All calculations were performed using Fortran 90 on an Dell server\(^{21}\) and on a Sun server\(^{22}\), both using double-precision floating-points numbers\(^{23}\).

In order to improve the performance, we used the conjugate gradient method for a sparse system\(^{24}\) to solve (29) and developed some specific subroutines\(^{25}\) to calculate \(\mathbf{D}\left[\mathbf{L}(y^{(m)}, \mu^{(m)})\right]\) and \(\mathbf{H}\left[\mathbf{L}(y^{(m)}, \mu^{(m)})\right]\) using sparse matrices in COO format (coordinate format).

To illustrate the profit in using sparse matrices, figures (9) and (10) display the form of matrices \(\mathbf{D}[j(g(y))]\) and \(\mathbf{H}[j(g(y))]\), respectively.

![Figure 9 - Form of D[j(g(y))] for n_l = n_h = 2.](image)

\(^{20}\)Note that in this specification \(\mathbf{H}[e(g)]\), the Hessian of \(e\), is a null \((30(n_l + n_h)^2 X 5(n_l + n_h))\) matrix.

\(^{21}\)Intel Pentium 4 1.70GHz and Intel Fortran Compiler 7.0.

\(^{22}\)Sun Fire 6800, 20 SPARC III 750 MHz processors, Sun Fortran 90 Compiler and Sun Performance Library (BLAS, SCALAPACK)

\(^{23}\)We also ran the code using quadruple-precision floating-point and confirmed the extremely accuracy of the results.

\(^{24}\)This subroutine was taken in Numerical Recipes in Fortran 90.

\(^{25}\)The subroutines developed were product, sum, kroncker products and transpose of sparse matrices and conversions from dense to sparse matrix and vice versa.
4.2 Persistence

In order to capture the role played by persistence, in this section we look at optimum in CF with no monetary policy improvement, \( n_l = n_h = 1 \), on a particular point in the parameter space, i.e., \( S = 3 \), \( B = 0.50 \), \( \beta = 0.79 \), \( \gamma_l = 3.0 \), \( \gamma_h = 2.5 \), over a grid of \( \pi, \psi \in \{0.01, 0.25, 0.50, 0.75, 0.99\} \).

We now define interest rates in period state low when last period was high as \( r_{<l,h>} \equiv \frac{y^n_{<l,1>}}{y^n_{<l,h>}} - 1 \) and interest rates in period state low when last period was low as \( r_{<l,l>} \equiv \frac{y^n_{<l,0>}}{y^n_{<l,l>}} - 1 \). In the same way, we define interest rates in period state high when last period was low as \( r_{<h,l>} \equiv \frac{y^n_{<h,1>}}{y^n_{<h,l>}} - 1 \) and interest rates in period state high when last period was high as \( r_{<h,h>} \equiv \frac{y^n_{<h,0>}}{y^n_{<h,h>}} - 1 \).

Table (4) displays the computed values of output relative to \( y^*_{l} \) and \( y^*_{h} \), in percentage points, for that particular point when shocks are independent\(^{26} \), i.e., \( \pi + \psi = 1 \).

\(^{26}\)In this case, invariant distribution of \( P \) is \( \varphi = [ \pi \ \psi ] \).
Table 4 - Percent of relative output for $B = 0.5$, $\beta = 0.79$, $\gamma_l = 3.0$, $\gamma_h = 2.5$ and $S = 3$

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>0.01</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi$</td>
<td>0.99</td>
<td>0.75</td>
<td>0.50</td>
<td>0.25</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table (5) displays interest rates derived from optimum in table (4).

As one can note, only when shocks are independent and all nonbank production constraints are binding, we found interest rates as function of current state, i.e., past information has no role in interest rates.

Table 5 - Interest rates (%) for $B = 0.5$, $\beta = 0.79$, $\gamma_l = 3.0$, $\gamma_h = 2.5$ and $S = 3$

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>0.01</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi$</td>
<td>0.99</td>
<td>0.75</td>
<td>0.50</td>
<td>0.25</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table (6) displays the computed values of output relative to $y^*_l$ and $y^*_h$, in percentage points, for that particular point when invariant distribution of $P$ is hold fixed at $\varphi = 0.5$, i.e., $\pi = \psi$. 

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Table 6 - Percent of relative output for $B = 0.5$, $\beta = 0.79$, $\gamma_l = 3.0$, $\gamma_h = 2.5$ and $S = 3$

<table>
<thead>
<tr>
<th></th>
<th>0.01</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>0.01</td>
<td>0.25</td>
<td>0.50</td>
<td>0.75</td>
<td>0.99</td>
</tr>
<tr>
<td>$\psi$</td>
<td>0.01</td>
<td>0.25</td>
<td>0.50</td>
<td>0.75</td>
<td>0.99</td>
</tr>
<tr>
<td>$y_{bb_{&lt;l&gt;}}^b$</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
</tr>
<tr>
<td>$y_{bb_{&lt;h&gt;}}^b$</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
</tr>
<tr>
<td>$y_{bn_{&lt;l&gt;}}^b$</td>
<td>89.3783</td>
<td>90.1835</td>
<td>91.7414</td>
<td>95.0001</td>
<td>99.7770</td>
</tr>
<tr>
<td>$y_{bn_{&lt;h&gt;}}^b$</td>
<td>92.7732</td>
<td>91.8701</td>
<td>90.0896</td>
<td>86.0945</td>
<td>79.4354</td>
</tr>
<tr>
<td>$y_{bn_{&lt;l&gt;}}^n$</td>
<td>108.0425</td>
<td>107.5736</td>
<td>106.6134</td>
<td>104.3468</td>
<td>100.2215</td>
</tr>
<tr>
<td>$y_{bn_{&lt;h&gt;}}^n$</td>
<td>106.1187</td>
<td>106.7534</td>
<td>107.9361</td>
<td>110.2986</td>
<td>113.5151</td>
</tr>
<tr>
<td>$y_{bn_{&lt;l&gt;}}^n$</td>
<td>102.4802</td>
<td>103.1965</td>
<td>104.2583</td>
<td>104.9999</td>
<td>100.2230</td>
</tr>
<tr>
<td>$y_{bn_{&lt;h&gt;}}^n$</td>
<td>76.6543</td>
<td>76.2870</td>
<td>75.8830</td>
<td>76.0764</td>
<td>80.4265</td>
</tr>
<tr>
<td>$y_{bn_{&lt;l&gt;}}^n$</td>
<td>102.4802</td>
<td>103.1965</td>
<td>104.2583</td>
<td>105.4163</td>
<td>100.4445</td>
</tr>
<tr>
<td>$y_{bn_{&lt;h&gt;}}^n$</td>
<td>76.6543</td>
<td>76.2870</td>
<td>75.8830</td>
<td>76.0764</td>
<td>80.4265</td>
</tr>
</tbody>
</table>

Table (7) displays interest rates derived from optimum in Table (6).

As one can note, when shocks are correlated, we found interest rates as function of current state as well as the previous one, i.e., past information has a role in interest rates.

This last result hugely contrast with Lucas (1990) findings. Lucas made simulations of an economy in which money was required for asset transaction as well as for transaction in goods. In that cash-in-advance model, government open-market operations induce liquidity effects that affect interest rate behavior. So, he used changes in the supplies of securities (made by government) as state governed by a Markov process

$$P_L = \begin{bmatrix} \kappa & 1 - \kappa \\ 1 - \kappa & \kappa \end{bmatrix},$$

assumed that lump sum transfers would maintain the money supply at a constant level and concluded:

"I found these simulations informative, in an unexpected direction. If one were to apply a model of this type to explaining or predicting actual short-term interest rate series, one would do very well simply by calculating the...

\footnote{Tables (I) and (II) at pages 259-60, in Lucas (1990).}

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constant equilibrium z-value\textsuperscript{28} for i.i.d. case studied in Section 4, and assuming it holds for any time pattern of the shocks. The cash allocation is so insensitive to advance information on bond issues, even when this information is very sharp compared to what one would ever see in practice, that these information effects can as well be ignored. Perhaps one can think of shock process where this would not be the case, but I was not able to do so."

\begin{table}[h]
\centering
\begin{tabular}{|c|ccccc|}
\hline
\pi & 0.01 & 0.25 & 0.50 & 0.75 & 0.99 \\
\hline
\psi & 0.01 & 0.25 & 0.50 & 0.75 & 0.99 \\
\hline
\text{r}_{<l,h>} & 2.59 & 2.63 & 2.26 & -0.17 & -9.30 \\
\text{r}_{<l,l>} & 5.43 & 4.24 & 2.26 & -0.62 & 0.90 \\
\text{r}_{<h,l>} & 42.27 & 42.13 & 42.24 & 44.33 & 55.62 \\
\text{r}_{<h,h>} & 38.44 & 39.94 & 42.24 & 44.98 & 41.14 \\
\hline
\end{tabular}
\caption{Interest rates (%) for B = 0.5, \( \beta = 0.79 \), \( \gamma_l = 3.0 \), \( \gamma_h = 2.5 \) and \( S = 3 \) in CF with \( n_l = n_h = 1 \).}
\end{table}

4.3 Monetary policy improvement

In this section, we look at optimum in CF with monetary policy improvement on the same particular point in the parameter space as in the last section, i.e., \( S = 3 \), \( B = 0.50 \), \( \beta = 0.79 \), \( \gamma_l = 3.0 \), \( \gamma_h = 2.5 \), over a grid of \( \pi, \psi \in \{0.01, 0.25, 0.50, 0.75, 0.99\} \).

We now define interest rates in first subperiod state low (\( l_1 \)) when last state was subperiod high (\( h_n \)) as
\[
\text{r}_{<l_1,h_n>} = \frac{\text{y}_{b_{<l_1>}}^{h_n}}{\text{y}_{b_{<h_n>}}} - 1, \text{ for } n \in \{1, 2, ..., n_h\}
\]
and interest rates in subperiod state low (\( l_n \)) when last state was the previous subperiod low (\( l_{n-1} \)) as
\[
\text{r}_{<l_n,l_{n-1}>} = \frac{\text{y}_{b_{<l_n>}}^{h_{n-1}}}{\text{y}_{b_{<l_{n-1}>}}} - 1, \text{ for } n \in \{2, 3, ..., n_l\}
\]

In the same way, we define interest rates in first subperiod state high (\( h_1 \)) when last state was subperiod low (\( l_n \)) as
\[
\text{r}_{<h_1,l_n>} = \frac{\text{y}_{b_{<h_1>}}^{l_n}}{\text{y}_{b_{<l_n>}}} - 1, \text{ for } n \in \{1, 2, ..., n_l\}
\]
and interest rates in subperiod state high (\( h_{n-1} \)) when last state was the previous subperiod high (\( h_{n-1} \)) as
\[
\text{r}_{<h_{n-1},h_{n-1}>} = \frac{\text{y}_{b_{<h_{n-1}>}}^{h_{n-2}}}{\text{y}_{b_{<h_{n-1}>}}} - 1, \text{ for } n \in \{2, ..., n_h\}
\]

We initiate the analysis by making \( n_l = 1 \) and \( n_h = 5 \), to capture the effects of adding subperiods in period state high.

Table (8) contains the computed values of output relative to \( y_l^* \) and \( y_h^* \), in percentage points, for the

\textsuperscript{28}The control variable \( z \) express the cash engaged in securitie trading.
particular point when $\pi$ is hold fixed at 0.25, with different values for persistence $\psi$.

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>0.25</th>
<th>0.25</th>
<th>0.25</th>
<th>0.25</th>
<th>0.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi$</td>
<td>0.01</td>
<td>0.25</td>
<td>0.50</td>
<td>0.75</td>
<td>0.99</td>
</tr>
<tr>
<td>$y_{ln}^1_{&lt;l1}&gt;$</td>
<td>91.4708</td>
<td>90.2941</td>
<td>88.8965</td>
<td>87.3057</td>
<td>85.5310</td>
</tr>
<tr>
<td>$y_{ln}^1_{&lt;h1}&gt;$</td>
<td>94.2444</td>
<td>93.5642</td>
<td>92.7975</td>
<td>91.9647</td>
<td>91.0638</td>
</tr>
<tr>
<td>$y_{ln}^1_{&lt;b1}&gt;$</td>
<td>88.1996</td>
<td>87.7610</td>
<td>87.3263</td>
<td>86.9003</td>
<td>86.4430</td>
</tr>
<tr>
<td>$y_{ln}^1_{&lt;h2}&gt;$</td>
<td>83.2700</td>
<td>83.2417</td>
<td>83.3153</td>
<td>83.4521</td>
<td>83.5582</td>
</tr>
<tr>
<td>$y_{ln}^1_{&lt;h3}&gt;$</td>
<td>79.5735</td>
<td>79.8537</td>
<td>80.5160</td>
<td>81.2475</td>
<td>81.9055</td>
</tr>
<tr>
<td>$y_{ln}^1_{&lt;h4}&gt;$</td>
<td>76.3030</td>
<td>76.8538</td>
<td>77.6150</td>
<td>78.4635</td>
<td>79.2305</td>
</tr>
<tr>
<td>$y_{ln}^1_{&lt;l2}&gt;$</td>
<td>106.7855</td>
<td>107.5078</td>
<td>108.3146</td>
<td>109.1711</td>
<td>110.0561</td>
</tr>
<tr>
<td>$y_{ln}^1_{&lt;h2}&gt;$</td>
<td>105.0306</td>
<td>105.5424</td>
<td>106.1014</td>
<td>106.6880</td>
<td>107.2998</td>
</tr>
<tr>
<td>$y_{ln}^1_{&lt;h3}&gt;$</td>
<td>109.1006</td>
<td>109.3584</td>
<td>109.6096</td>
<td>109.8516</td>
<td>110.1070</td>
</tr>
<tr>
<td>$y_{ln}^1_{&lt;h4}&gt;$</td>
<td>111.7605</td>
<td>111.7744</td>
<td>111.7383</td>
<td>111.6709</td>
<td>111.6184</td>
</tr>
<tr>
<td>$y_{ln}^1_{&lt;l3}&gt;$</td>
<td>113.5484</td>
<td>113.3354</td>
<td>113.0452</td>
<td>112.7165</td>
<td>112.4132</td>
</tr>
<tr>
<td>$y_{ln}^1_{&lt;l4}&gt;$</td>
<td>114.7801</td>
<td>114.5676</td>
<td>114.2672</td>
<td>113.9226</td>
<td>113.6022</td>
</tr>
<tr>
<td>$y_{ln}^{nB}<em>{0&lt;l1}&gt; = y</em>{01}^{nn}_{&lt;l1}&gt;$</td>
<td>102.3762</td>
<td>102.8740</td>
<td>103.5312</td>
<td>104.3139</td>
<td>105.1536</td>
</tr>
<tr>
<td>$y_{ln}^{nB}<em>{0&lt;h1}&gt; = y</em>{01}^{nn}_{&lt;h1}&gt;$</td>
<td>76.2533</td>
<td>76.4963</td>
<td>76.8750</td>
<td>77.3598</td>
<td>77.8711</td>
</tr>
<tr>
<td>$y_{ln}^{nB}<em>{0&lt;h2}&gt; = y</em>{01}^{nn}_{&lt;h2}&gt;$</td>
<td>76.2653</td>
<td>76.8155</td>
<td>77.5358</td>
<td>78.3601</td>
<td>79.1794</td>
</tr>
<tr>
<td>$y_{ln}^{nB}<em>{0&lt;h3}&gt; = y</em>{01}^{nn}_{&lt;h3}&gt;$</td>
<td>76.2753</td>
<td>77.0651</td>
<td>78.0226</td>
<td>79.0596</td>
<td>80.0583</td>
</tr>
<tr>
<td>$y_{ln}^{nB}<em>{0&lt;h4}&gt; = y</em>{01}^{nn}_{&lt;h4}&gt;$</td>
<td>76.2835</td>
<td>77.2634</td>
<td>78.4022</td>
<td>79.6000</td>
<td>80.7371</td>
</tr>
<tr>
<td>$y_{ln}^{nB}<em>{0&lt;h5}&gt; = y</em>{01}^{nn}_{&lt;h5}&gt;$</td>
<td>76.2835</td>
<td>77.2634</td>
<td>78.4022</td>
<td>79.6000</td>
<td>80.7371</td>
</tr>
</tbody>
</table>

Table 8 - Percent of relative output for $B = 0.5$, $\beta = 0.79$, $\gamma_l = 3.0$, $\gamma_h = 2.5$ and $S = 3$

in CF with $n_l = 1$, $n_h = 5$ and $\pi = 0.25$. All production between banks are 100.0000.

Figure (11) shows interest rates in subperiods of kind high, derived from optimum in table (8). Interest rates converge to a level that depends on persistences $\pi$ and $\psi$.

As one can note in figure (11), we found the path of interest as function of current state\textsuperscript{29} and also highly

\textsuperscript{29} As we will see further, when we make $n_l > 1$, past information is also influential.
function of persistence $\psi$. In particular, the greater the probability to change to first state low, i.e., the less is $\psi$, the greater is interest rate.

We also note that, when $\psi \rightarrow 1$, interest rates in subperiod state high $r_{<h_n,>}\approx$ oscillate around $40.61\%$. This is the level when economy is forever in state high. This value can be extracted from table (2) in the column labelled C.

![Interest rates in period state high](image)

**Figure 11 - Graph of interest rates (%) for $B = 0.5$, $\beta = 0.79$, $\gamma_l = 3.0$, $\gamma_h = 2.5$ and $S = 3$ in CF with $n_l = 1$, $n_h = 5$ and $\pi = 0.25$, for $\psi \in \{0.01, 0.25, 0.50, 0.75, 0.99\}$.

Now, make $n_l = 5$ and $n_h = 1$, to capture the effects of adding subperiods in period state low.

Figure (12) shows interest rates in subperiods of kind low, derived from optimum in the particular point, when $\psi$ is hold fixed at $\psi = 0.25$, with different values for persistence $\pi$. Interest rates always converge to zero, which is the level when economy is forever in state low. This value can be extracted from table (1) in the column labelled I.
This asymmetry between states low and high reflects the fact that when economy is in state high, bank’s sector cares about the possibility of changing to a different state, i.e., bank acts forward-looking.

On the other hand, when economy is in state low, bank’s sector acts backward-looking.

We can now make \( n_l = 5 \) and \( n_h = 10 \), where interest is a function of current substate, as well as past substate, when state changes\(^3\).

Following Katzman, Kennan and Wallace (2001), we can define total output and a price index for this economy. Let \( y_{<p>} \) and \( P_{<p>} \) denote, respectively, this aggregates in period state \( p \).

For our mechanism, total output is the appropriate sum of outputs over meetings:

\[
S \ y_{<p>} = x_b x_y y_{<p>} + x_h x_0 y_{<p>} + x_y x_1 y_{<p>} + x_0 x_1 y_0 y_{<p>} + x_0 x_1 y_1 y_{<p>}
\]

for \( p \in \{ l_1, ..., l_n, h_1, ..., h_n \} \).

We take \( P_{<p>} \) to be the total output deflator; namely, total nominal output, denoted \( Y_{<p>} \), divided by \( y_{<p>} \). Total nominal output is a weighted sum of the number of meetings where trade occurs. That is,

\[
S \ Y_{<p>} = x_b x_y + x_h x_0 + x_y x_1 + x_0 x_1 + x_0 x_1 = B + \left( \frac{1-B^2}{4} \right).
\]

Figure 12 - Graph of interest rates (\%) for \( B = 0.5, \beta = 0.79, \gamma_l = 3.0, \gamma_h = 2.5 \) and \( S = 3 \) in CF with \( n_l = 5, n_h = 1 \) and \( \psi = 0.25 \), for \( \pi \in \{0.01, 0.25, 0.50, 0.75, 0.99\} \).
As a way to highlight some of the effects, figures (13) and (14) depict total output, while figures (15) and (16) show the price index, both in states low and high, respectively, when shocks are independent.

Figure 13 - Graph of total output for $B = 0.5, \beta = 0.79, \gamma_l = 3.0, \gamma_h = 2.5$ and $S = 3$ in CF with $n_l = 5$ and $n_h = 10$ when shocks are independent.

Figure 14 - Graph of total output for $B = 0.5, \beta = 0.79, \gamma_l = 3.0, \gamma_h = 2.5$ and $S = 3$ in CF with $n_l = 5$ and $n_h = 10$ when shocks are independent.
Figure 15 - Graph of price index for \( B = 0.5, \beta = 0.79, \gamma_l = 3.0, \gamma_h = 2.5 \) and \( S = 3 \) in CF with \( n_l = 5 \) and \( n_h = 10 \) when shocks are independent.

As one can see, total output and price index are negatively correlated. The same result was found in Katzman, Kennan and Wallace (2001) when all agents are informed about real shocks.

Another interesting result is that total output in all subperiods in state low is greater than the amount traded if in all single-coincidence meeting \( y_l^* \) were produced.
5 Concentration of the bank sector in a monetary model

In this chapter, we generalize the Cavalcanti-Wallace’s model (CW) in a different fashion. This generalization allows for bank competition, in the sense that banks can choose previously between two different banking nets. Moreover, here banks are not asked to produce for nonbanks without a note, i.e., no gifts are allowed.

The environment and the definition of stationary and symmetric inside money allocations are quite analogous to CW, with some little changes.

First, there is a banking net that issues red notes, called red, that only accepts banks that trade in exchange of red notes and do not cooperate with another bank, unless the last is a red banking net member. There is another banking net that issues green notes, called green, that usually trades in exchange of green notes but is allowed to receive, hold and give red notes, only when necessary. Green banks do not cooperate with red banks either.

Second, all notes issued by a banking net are treated identically, but can be easily distinguished from those issued by the other net.

Third, banks must choose between nets ex-ante, i.e., once and for all.

Thus, we let production and consumption in a single-coincidence meeting be denoted $y_{ij}^{km} \in \mathbb{R}^+$ where the superscripts denote identity - $k,m \in \{R\text{(red banker)},G\text{(green banker)},n\text{(nonbanker)}\}$ where $k$, the first superscript, is the identity of the producer and $l$, the second, is that of the consumer - and the subscripts denote states for green bankers - $i,j \in \{\cdot,R\}$ and for nonbankers - $i,j \in \{0,G,R\}$ where $i$, the first subscript, is that of the producer and $j$, the second, is that of the consumer.

We let $x^k_i$ with $k \in \{R,G,n\}$, $i \in \{\cdot,R\}$ when $k = G$ and $i \in \{0,G,R\}$ when $k = n$, denote the fraction of each production-consumption specialization type who have identity $k$ and, in case of green bankers and nonbankers, who are in state $i$. So, these fractions satisfy:

$$x^R = B_R, \ x^G + x^G_R = B - B_R \text{ and } x^0_0 + x^G_0 + x^R_R = (1 - B)$$  \hspace{1cm} (30)

For $0 \leq B_R \leq B \leq 1$.

So, a particular group of stationary and symmetric inside money allocations derived from that particular
regulation becomes described as follows: in all single-coincidence meetings between nonbankers, there is production in exchange of a note if and only if that is consistent with preservation of note holdings in the meeting and the unit upper bound on holdings. A nonbanker that does not have any note produces for a bank and receives a note issued by the bank. A nonbanker that has a note produces for a bank, if and only if, she receives a different note. A red bank produces for a nonbanker, if the nonbanker has a red note, which the bank receives and destroys. A green bank produces for a nonbanker holding a green note, which the bank receives and destroys, and produces for a nonbanker holding a red note, which the bank receives and holds, unless the bank has a red note already. A green bank always produces to another green bank. If the last has a red note and the previous does not, the note switch hands. Moreover, a green bank produces for a red bank and receives a red note if and only if she does not hold a red note. Finally, a red bank always produces to another bank in his banking net. In all other meetings nothing happens.

This allocation is implementable if it satisfies some participation constraints. In order to express participation constraints in terms of discounted expected utilities, we first define $v^k_i$, the expected discounted utility for a person with identity $k$ who begins a period in state $i$.

The first constraints are restrictions on production by green bankers without a red note:

\[-y_{GG}^i + \beta v^G_i \geq 0\]  \hspace{1cm} (31)
\[-y_{GR}^i + \beta v^G_R \geq 0\]  \hspace{1cm} (32)
\[-y_{Gn}^i + \beta v^G_{Rn} \geq 0\]  \hspace{1cm} (33)
\[-y_{Gn}^i + \beta v^G_{Gn} \geq 0\]  \hspace{1cm} (34)
\[-y_{R}^i + \beta v^G_R \geq 0\]  \hspace{1cm} (35)

The next constraints are restrictions on production by green bankers with a red note:

\[-y_{GG}^i + \beta v^G_R \geq 0\]  \hspace{1cm} (36)
\[-y_{Gn}^i + \beta v^G_{Rn} \geq 0\]  \hspace{1cm} (37)
\[-y_{Gn}^i + \beta v^G_{Gn} \geq 0\]  \hspace{1cm} (38)
The next constraints are restrictions on production by red bankers:

\[-y_{RG}^R + \beta v_R^R \geq 0 \quad (39)\]
\[-y_{RR}^R + \beta v_R^R \geq 0 \quad (40)\]
\[-y_{Rn}^R + \beta v_R^R \geq 0 \quad (41)\]

The right-hand side of (31) - (41) are zero because the highest feasible punishment prevents banks from consuming after a defection.

We require that nonbankers have non negative gains from trade when they consume. The next constraints summarize this,

\[u (y_{OG}^n) + \beta v_0^n \geq \beta v_G^n \quad (42)\]
\[u (y_{OG}^n) + \beta v_0^n \geq \beta v_G^n \quad (43)\]
\[u (y_{RG}^n) + \beta v_0^n \geq \beta v_G^n \quad (44)\]
\[u (y_{OR}^n) + \beta v_0^n \geq \beta v_R^n \quad (45)\]
\[u (y_{GR}^n) + \beta v_0^n \geq \beta v_R^n \quad (46)\]
\[u (y_{RG}^n) + \beta v_0^n \geq \beta v_R^n \quad (47)\]
\[u (y_{OR}^n) + \beta v_0^n \geq \beta v_R^n \quad (48)\]
The next constraint requires that nonbankers have nonnegative gains from trade when they produce,

\[-y_{0G}^n + \beta v_G^0 \geq \beta v_0^G \quad (49)\]

\[-y_{0R}^n + \beta v_R^0 \geq \beta v_0^R \quad (50)\]

\[-y_0^G + \beta v_G^0 \geq \beta v_0^G \quad (51)\]

\[-y_0^G + \beta v_R^0 \geq \beta v_0^G \quad (52)\]

\[-y_0^R + \beta v_R^0 \geq \beta v_0^R \quad (53)\]

\[-y_{0G}^n + \beta v_G^0 \geq \beta v_0^G \quad (54)\]

\[-y_{0G}^n + \beta v_R^0 \geq \beta v_0^G \quad (55)\]

\[-y_{0R} + \beta v_R^0 \geq \beta v_0^R \quad (56)\]

We will also assume that a nonbanker does not freely dispose money. As a consequence, nonbankers who possess a red note will never produce until she consumes and gives up money.

The stationarity across states implies that the inflow into state 0 and G must equal the outflow from state 0 and G respectively, for nonbankers and the inflow into state R must equal the outflow from state R for green bankers, i.e.:

\[x_G^n [B - B_R] + x_R^n B_R = x_0^n B \quad (57)\]

\[x_0^n [B - B_R] + x_G^n x_R^n = x_0^n [B + x_G^n] \quad (58)\]

\[x_G^n [B_R + x_R^n] = x_R^n [x_G^n + B_R] \quad (59)\]

Equations (30) and (57)-(59) supply a system of nonlinear equations whose the solution is given in appendix B.

The stationarity also implies that we can write:

\[v_0^n = \frac{x_G^n}{S} (-y_{0G}^n) + \frac{x_R^n}{S} (-y_{0R}^n) + \frac{x_G^n}{S} (-y_0^G) + \frac{x_R^n}{S} (-y_0^R) + \]

\[+ \beta \left[ \left( \frac{x_G^n + x_G^n + x_R^n}{S} \right) v_G + \left( \frac{x_R^n + x_R^n}{S} \right) v_R + \left( 1 - \frac{x_G^n + x_G^n + x_R^n + x_R^n}{S} \right) v_0 \right] \quad (60)\]
\[
v^n_G = \frac{x^n_0}{S}u(y^n_G) + \frac{x^n_0}{S}u(y^{G_0}_G) + \frac{x^n_R}{S}u(y^n_{GR}) + \frac{x^n_R}{S}(-y^n_{GR}) + \frac{x^n_R}{S}(-y^n_G) + \\
+ \beta \left\{ \left( \frac{x^n_0 + x^n_G + x^n_R}{S} \right) v^n_0 + \left( \frac{x^n_R + x^n_G + x^n_R}{S} \right) v^n_R + \left( 1 - \frac{x^n_0 + x^n_G + x^n_R}{S} \right) v^n_G \right\},
\]

\[
v^n_R = \frac{x^n_0}{S}u(y^n_{0R}) + \frac{x^n_0}{S}u(y^{G_0}_{GR}) + \frac{x^n_0}{S}u(y^{G_0}_{RR}) + \frac{x^n_0}{S}u(y^{G_0}_R) + \\
+ \beta \left\{ \left( \frac{x^n_0 + x^n_R}{S} \right) v^n_0 + \left( \frac{x^n_R + x^n_G}{S} \right) v^n_G + \left( 1 - \frac{x^n_0 + x^n_R}{S} \right) v^n_R \right\}
\]

\[
v^G = \frac{x_n^u}{S}u(y^n_G) + \frac{x^n_0}{S}u(y^n_{GR}) + \frac{x^n_0}{S}u(y^n_{RR}) + \frac{x^n_0}{S}u(y^n_R) + \\
+ \beta \left\{ \left( \frac{x^n_R + x^n_G + x^n_R}{S} \right) v^n_G + \left( 1 - \frac{x^n_0 + x^n_R + x^n_R}{S} \right) v^n_R \right\}
\]

\[
v^n_R = \frac{x^n_0}{S}u(y^n_{0R}) + \frac{x^n_0}{S}u(y^n_{GR}) + \frac{x^n_0}{S}u(y^n_{RR}) + \frac{x^n_0}{S}u(y^n_R) + \\
+ \beta \left\{ \left( \frac{x^n_R + x^n_G + x^n_R}{S} \right) v^n_G + \left( 1 - \frac{x^n_0 + x^n_R + x^n_R}{S} \right) v^n_R \right\}
\]

Notice that for a given allocation, \( (60) \) up to \( (65) \) consist of 6 linear equations in all expected discounted utilities. Those equations have a unique solution. Among the ways to establish that is by a trivial contraction mapping argument.

The optimization problem is to maximize a welfare function over the set of implementable, stationary
and symmetric inside money allocations implied by the regulation. The representative-agent criterion is

\[ w = \sum_{k,i} x^k_i v^k_i \]  

(66)

where \( (k, i) \in \{(n, 0), (n, G), (n, R), (G, ), (G, R), (R, \cdot)\} \)

Another way to write the welfare criterion is, by equations (13) and (14)-(19),

\[ w = \sum_{k,m,i,j} x^k_i x^m_j S(1 - \beta) z(y^{km}_{ij}) \]  

(67)

where \( (k, m, i, j) \in \{(n, n, 0, G), \ldots, (R, G, \cdot, R)\} \), \( z(y^{km}_{ij}) = u(y^{mk}_{ji}) - y^{km}_{ij} \).

Hence, \( w \) can be understood as the present value expected in this economy.

**Proposition 2** The optimization problem over the set of implementable, stationary and symmetric inside money allocations implied by the regulation has a unique solution that is continuous in \( \theta = (B, B_R, \beta) \in \Theta = \{[0,1]^2 \times [0,1)\} \).

**Proof.** Appendix C. \( \blacksquare \)

From (67), it is easy to see that \( y^* \), the first-best level of output is such that \( z'(y^*) = 0 \).

From now on, our analysis and graphs will focus on the case when constraints do not bind at first-best. This is done so because this simple case, where everything goes as if goods were indivisible, suffices to reach the basic economic ideas we intend to show.

Since banks must choose ex-ante between nets, if a green bank observe ex-ante that, in average, the red sector is better off, it chooses to join the red banking net. Otherwise, a red bank could decide to join the green banking net.

Therefore, we can derive a movement rule for the banking sector, described as follows: a bank join the green banking net if the average pay off for the green sector is better off, i.e., \( RA(B, B_R) < 0 \), where

\[ RA(B, B_R) \equiv v^R - \frac{x^G v^G + x^G_R v^G_R}{x^G + x^G_R} \]  

(68)
In the same fashion, a join the red banking net if the average pay off for the red sector is better off, i.e.,
\[ RA(B, B_R) > 0. \]

A competitive equilibrium can be found when \( RA(B, B_R) = 0. \)

**Proposition 3** Suppose that constraints do not bind at first-best:

i) If \( B \in (0, 1) \) is sufficiently small, i.e., \( 0 < B < B \doteq 0.12405 \), and \( \frac{u(y^*)}{y^*} < \left( \frac{B+1}{2B} \right)^2 \), then 
\[ \lim_{B_R \to 0} RA(B, B_R) > 0 \text{ and } \lim_{B_R \to B^-} RA(B, B_R) < 0. \]

ii) If \( B \in (0, 1) \) is sufficiently large, i.e., \( 0.12405 < B < 1 \), and \( \frac{u(y^*)}{y^*} > \left( \frac{B+1}{2B} \right)^2 \), then 
\[ \lim_{B_R \to 0} RA(B, B_R) < 0 \text{ and } \lim_{B_R \to B^-} RA(B, B_R) > 0. \]

**Proof.** Appendix D.

Figures (17)-(20) display the graphs of relative average pay off, value functions and invariant distributions across states for \( B = 0.1 \) and \( B = 0.6 \). In figures (17) and (18) we get an stable equilibrium in bank competition for \( B = 0.1, \beta = 0.98, \frac{u(y^*)}{y^*} = 6.36 \) and \( S = 3 \), described by \( \{ x_n^0 = 0.3049, x^G_n = 0.1547, x^R_R = 0.4403, x^G = 0.0140, x^G_R = 0.0334, x^R = 0.0526 \} \), 
\( \{ y^*_m^k = y^*, \forall (k, m, i, j) \in \{ (n, n, 0, G), \ldots, (R, G, G, R) \} \} \).

![Graphs of relative average pay off (RA) and value functions](image)

Figure 17 - Graphs of relative average pay off (RA) and value functions for \( B = 0.1, \beta = 0.98, \frac{u(y^*)}{y^*} = 6.36 \) and \( S = 3 \) over \( B_R \in [0, 0.1] \).
While in figures (19) and (20) we get an unstable equilibrium in bank competition for $B = 0.6$, $\beta = 0.98$, $u(y^\ast) = 6.36$ and $S = 3$, described by

$$\left\{ x_n^0 = 0.1342, x_n^G = 0.0826, x_R^1 = 0.1833, x_G^G = 0.1296, x_R^G = 0.1631, x_R^R = 0.3074 \right\},$$

$$\left\{ y_{mk}^{mi} = y^\ast, \forall (k, m, i, j) \in \{(n, n, 0, G), \ldots, (R, G, \cdot, R)\} \right\}.$$
for $B = 0.1$, $\beta = 0.98$, $\frac{u(y^*)}{y^*} = 6.36$ and $S = 3$ over $B_r \in [0, 0.6]$.

We can plot (figure 21) the total of notes in the economy, $M = x^n_G + x^n_R + x^G_R$, for different values of the bank sector $B$, changing the amount of red banks in it, $B_r \in [0, B]$.

Denote $\rho^G = \frac{x^G}{y^* - x_R}$ and $\rho^G_R = \frac{x^G_R}{x^G_R - x_R}$. Differentiating (68) with respect to $B_R$, produces:
\[
\frac{\partial R(A, B_R)}{\partial B_R} = \frac{\partial v_R}{\partial B_R} - \left[ \rho_G \frac{\partial v_G}{\partial B_R} + \rho_R \frac{\partial v_R}{\partial B_R} + \frac{\partial \rho_G}{\partial B_R} v_G + \frac{\partial \rho_R}{\partial B_R} v_R \right]
\] (69)

Since \( \frac{\partial \rho_G}{\partial B_R} = -\frac{\partial \rho_R}{\partial B_R} \),

\[
\frac{\partial R(A, B_R)}{\partial B_R} = \frac{\partial v_R}{\partial B_R} - \left[ \rho_G \frac{\partial v_G}{\partial B_R} + \rho_R \frac{\partial v_R}{\partial B_R} + \frac{\partial \rho_G}{\partial B_R} \left( v_R - v_G \right) \right]
\] (70)

Red bankers destroy all notes, while green bankers keep red notes. When \( B \) is small enough, red bankers entrance turns the total of notes down. Therefore, nonbanking liabilities for the green banking net is reduced and the number of potential nonbank producers, i.e., nonbankers with no note, increases. Provided the benefit inside the green net, i.e., the welfare derived from production between green bankers is not so large, all green bankers can improve. This explain the conditions on proposition 3i. Moreover, as a consequence, the term in square brackets on the right-hand side of equation (70) is positive and \( \frac{\partial R(A, B_R)}{\partial B_R} < 0 \) near the stable competitive equilibrium. We can connect this to a money externality.

When \( B \) is large enough and the benefit inside the green net is large, green bankers are worse off when the red banking net increases. This explain the conditions on proposition 3ii. As a consequence, the term in square brackets on the right-hand side of equation (70) is negative and \( \frac{\partial R(A, B_R)}{\partial B_R} > 0 \) near the unstable competitive equilibrium. Otherwise, we can connect this to a credit externality.

This credit externality has already been described in Cavalcanti and Wallace. While the money externality effect is something new in a random-matching banking model.

Note that from figure (19), it is easy to identify two more stable equilibria. If the total of green bankers collapse to zero, we get an pure inside money with red notes, simmilar to CW. On the other hand, if the total of red bankers collapse to zero, we get an mix of red outside money and green inside money banking sector that accepts outside money.

We also found a case where there are two stable and one unstable equilibrium in banking competition and a case where there is only one stable equilibrium, the mix one or the pure inside money with red notes.

Note that if we consider that no one can be monitored in the economy, the mix equilibrium collapses to
an outside money for green and red notes.

Now we can proof the existence of stable equilibria and unstable equilibria.

**Proposition 4** If $B \in (0,1)$ is sufficiently small and $\beta \in (0,1)$ and $\frac{u(y^*)}{y^*}$ are sufficiently large, then in a neighborhood of an first-best equilibrium: $v^n_R > v^n_G > v^0_0 > 0$, $v^R > 0$, $v^G_R > v^G > 0$ and constraints do not bind.

**Proof.** Appendix E. ■

**Proposition 5** If $B \in (0,1)$ is sufficiently small, $\beta \in (0,1)$ is sufficiently large and $3 < \frac{u(y^*)}{y^*}$, then there are stable equilibria in bank competition.

**Proof.** If $B \rightarrow 0$, $\beta \rightarrow 1$ and $3 < \frac{u(y^*)}{y^*}$, there exist stable equilibria in bank competition, since by proposition 4, $v^n_R > v^n_G > v^0_0 > 0$, $v^R > 0$, $v^G_R > v^G > 0$ and constraints do not bind at first-best. Once constraints do not bind at first-best and $(\frac{B+1}{2B})^2$ becomes large, by proposition 3i, $RA(B,0+) > 0$ and $RA(B,B-) < 0$. ■

**Proposition 6** If $B \in (0,1)$, $\beta \in (0,1)$ and $\frac{u(y^*)}{y^*}$ are sufficiently large, then in a neighborhood of an first-best equilibrium: $v^n_R > v^n_G > v^0_0 > 0$, $v^R > 0$, $v^G_R > v^G > 0$ and constraints do not bind.

**Proof.** Appendix F. ■

**Proposition 7** If $B \in (0,1)$ and $\beta \in (0,1)$ are sufficiently large and $3 < \frac{u(y^*)}{y^*}$, then there are unstable equilibria in bank competition.

**Proof.** If $B \rightarrow 1$, $\beta \rightarrow 1$ and $3 < \frac{u(y^*)}{y^*}$, there exist stable equilibria in bank competition, since by proposition 6, $v^n_R > v^n_G > v^0_0 > 0$, $v^R > 0$, $v^G_R > v^G > 0$ and constraints do not bind at first-best. Once constraints do not bind at first-best and $(\frac{B+1}{2B})^2$ goes to 1, by proposition 3ii, $RA(B,0+) < 0$ and $RA(B,B-) > 0$. ■

In appendix G, we discribe the case where green banks ignore red notes.
Another interesting result is that for low values of $B$, society welfare is highly influenced by the capacity of distinction between two colors of notes, a feature that is present in Cavalcanti (2000). Even when some individuals already hold a green note, they must be wishing to produce for a more valuable object. As a consequence, trades increases. Figures (22)-(24) illustrates this point. We can note from figures (22) and (23) that credit externality causes monotonicity and symmetry on welfare, while from figure (24) one can see that money externality causes a lack of monotonicity on welfare.

Figure 22 - Graph of $w$ over $B_R \in [0, B - B_g]$ with $B_g$ fixed for $\beta = 0.98$, $\frac{u(y^*)}{y^*} = 6.36$ and $S = 3$. 

for $\beta = 0.98$, $\frac{u(y^*)}{y^*} = 6.36$ and $S = 3$. 

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Figure 23 - Graph of $w$ over $B_R \in [0, B - B_R]$ with $B_R$ fixed for $\beta = 0.98$, $\frac{u(y^*)}{y^*} = 6.36$ and $S = 3$.

Figure 24 - Graph of $w$ for $B$ low and for $\beta = 0.98$, $\frac{u(y^*)}{y^*} = 6.36$ and $S = 3$. 
6 Concluding remarks

The random-matching models of money can be seen the replacement of the overlapping generations model of money. The latter have nice attributes such as the general equilibrium structure and the absence of money as an argument of utility functions or production functions. Many economists were attracted to the overlapping generations model because they seemed to provide a sound theoretical basis for the development of a theory of fiat, outside money. After some time, critics of the model pointed out that outside money is not a robust institution in that model because other assets such as bonds and capital could replace fiat money and often promote Pareto optimality. Even policy interventions, such as a social security arrangement, which is in principle an implementable mechanism in that model, could drive money out of use. Some economists conjectures that the tenuousness of monetary equilibria in the overlapping generations model happens because money does not perform the role of medium of exchange in that model. As a result, the model was destined to be silent on many monetary phenomena that have to do with media of exchange. Tobin (1980), for instance, criticizes the defense by Wallace (1980) of the overlapping generations model on the grounds that, in that model, money has a constant velocity, and would never address questions about the creation of private currency and other kinds of inside money that may substitute outside money.

The random-matching model of money is a sharp departure from Walrasian models. Too sharp for many economists, that see in the model an excessively puristic attempt to describe the medium-of-exchange role of money without concern with applied questions. This criticism has achieved some influence in the profession as a result of the extreme assumptions that give rise to a central role for money in the model. The need of such assumptions are, however, not surprising. Since attempts to introduce money in the framework of competitive general equilibrium led to failure, there was a need to bypass the standard, centralized, trade arrangements implicit in the Walrasian analysis. Kiyotaki and Wright (1989) achieved considerable success by assuming that all individuals in the model are anonymous, and because they cannot be monitored, all individual histories are private information. Since trade takes place in pairs, during random meetings, no useful information becomes public knowledge after meetings, so that no credit arrangement can take place. As a result, durable fiat money (or even durable goods) can perform the medium of exchange role under the assumption that people specialize in the goods they derive utility from and in the goods that they can
produce. The specialization of individuals of different types in producing and consuming different goods generates an absence of the so-called double coincidence of wants that has long been recognized as important by classic writers such as Smith and Jevons.

The assumptions that all trade takes place during random meetings, and that only outside money (ignoring commodity money) can facilitate trade, are considered extreme for some purposes. Moreover, Kiyotaki and Wright (1989) have to assume that people can only hold either zero or one unit of money, that is assumed to be indivisible. The zero-one money holdings assumption avoids the complication of describing an endogenous distribution of money in steady states. Despite these extreme assumptions, many monetary questions can now be addressed with the random-matching model, such as the effects of changing the quantity of outside money, and producing Phillips-curve observations in the spirit of Lucas (1972), or ranking equilibria in terms of welfare when multiple currencies coexist (Kiyotaki, Matsuyama and Matsui, 1993). Recent research is attempting to reduce the zero-one restriction.

Although Kiyotaki and Wight (1989) deliver a model in which money sometimes is not spent, so that the velocity of money is endogenous, it is difficult to say it addresses the criticism formulated by Tobin. Many economists thought that the Kiyotaki-Wright model could only describe outside money, that is, the use of monetary assets in fixed supply, because the assumption of anonymity seemed necessary for the essentiality of money in the model. Moreover, with histories totally private, there would never be room for credit arrangements in general, and for inside money in particular. Kocherlakota and Wallace (1998) are possibly the first authors to point out to a different direction by applying mechanism design to the environment of Kiyotaky and Wright under the assumption that the history of trade of all individuals, from one date to the next, can become public knowledge with a positive probability. They showed that when that probability approaches one, an optimal allocation has all individuals giving gifts during random meetings, and when it approaches zero, the optimum has money trading for goods as in the original Kiyotaki and Wright model.

In a different venue, Cavalcanti, Erosa, and Temzelides (1999) modified the Kiyotaki and Wright model by assuming that a fraction of the population is monitored, and can moreover issue private notes that are cleared against reserves monitored by a central bank or clearing house entity. Since building reserves required

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[31] The term inside money, up to this point, has only been used to describe arrangements involving borrowing and lending, but without reference to any monetary aspect of the means of payments, if any, used for closing transactions.
accepting and destroying notes issued by others, Cavalcanti, Erosa, and Temzelides showed that a stable monetary equilibria with private notes exists if the discount factor is sufficiently high.

Cavalcanti and Wallace (1999) applied mechanism design to an environment similar in spirit to the banking model of Cavalcanti, Erosa, and Temzelides. The model of Cavalcanti and Wallace (1999), CW for short, forms the basis of the models of this thesis, but with a different motivation. The main point of CW is the comparison of outside money with inside money since the latter term has long been used in the literature without reference to any monetary properties. Ongoing questions, that date back to proposals of regulating money according to the currency (Friedman) and banking (Smith) schools of thought, were never phrased in terms of modeled properties of alternative means of payments. Is inside money more liquid than outside money? Can it achieve a higher welfare? How implementable allocations differ from inside-money regimes to the outside-money ones?

The recipe chosen by CW to address these questions can be described roughly as follows. They assumed that a fraction $B \in (0, 1)$ of the population is monitored perfectly (no need use reserves for that purpose) and are called bankers. The assumed that the other fraction, of measure $1 - B$, are totally anonymous. Then they describe symmetric and stationary allocations that satisfy participation constraints under two types of regulation environments. In the first, bankers are permitted to issue and destroy money according to the recommendation of possibly optimal plans. Implementable allocations in this set are called inside-money allocations. In the second, bankers are never permitted to issue inside money, so that the stock of money in circulation is transferred as in the Kiyotaki and Wright model. Implementable allocations in this second set are naturally called outside-money allocations. The main result in CW is that the latter set of allocations is a strict subset of the former. They show that any outside-money allocation can be duplicated with inside money as long as the planner recommends a particular pattern of creation and destruction of bank notes, and that kind of regulation is feasible under the monitoring assumptions adopted in the paper.

The strict-subset result of Cavalcanti and Wallace confirms the robustness of banking equilibria of Cavalcanti, Erosa and Temzelides (1999), but has implications of their own. The strict-subset result is derived without the need to solve for an optima, since comparing the sets of allocations from which the optimum must be chosen suffices for the argument. The result also says that, contrary to what many
economists thought, the Kiyotaki-Wright approach leads to a model of inside money, for even for $B$ arbitrarily small, inside money dominates outside money in welfare terms, and thus outside money is actually tenuous in random-matching models. The welfare implication have also deep implications for the kind of policy recommendations of currency-school proponents such as Friedman and others that propose strong banking regulations. The mechanism-design approach of $CW$ may also allow comparisons of money models with baking models such as that of Diamond and Dybvig (1983) that also solve planner problems.

The results in Kocherlakota and Wallace (1998) and in Cavalcanti and Wallace (1999) offer a clean language and approach for discussing some fundamental properties of monetary systems, but there are implications about the theory of banking that $CW$, or even Cavalcanti, Erosa, and Temzelides (1999) ignore. In the latter, banks compete with each other for reserves, but they do not form networks that take advantage of credit externalities. There is no cooperation among subsets of banks in Cavalcanti, Erosa, and Temzelides (1999), but also, there is no attempt to regulate banks in desirable ways in that model, except by imposing lower bounds on reserves so that a stable equilibrium attains. By contrast, there is no bank competition in $CW$. Banks in $CW$ form one large network and cooperate among themselves optimally. Moreover, the only effort to describe optima in detail in $CW$ takes place after the derivation of the strict-subset result and for the particular case of the parameter space for which participation constraints do not bind.

In this thesis we modify the $CW$ in two major ways. First, we ask the question of whether two coalitions of banks, issuing money with different rates of return, can compete and coexist in a stable stationary equilibrium. We think that the $CW$ can be amended to address bank competition and the issue of private instruments with different velocities, in ways that would take care of the concerns expressed by Tobin (1980) regarding Walrasian models of money. We find that the answer to this question depends crucially to the aggregate measure of banks $B$, a parameter with the same interpretation as that given by $CW$. Although the strict subset result of $CW$ is invariant to value of $B$, the coexistence of networks depends on $B$ in our model because high values $B$ increases credit externalities leading to the monopoly in currency issue. By contrast, low values of $B$ brings to the scene the bank liabilities in the form of competing currencies in circulation: large networks have large quantities of money in circulation redeemable in services that cause disutilities. The relationship between $B$ and the equilibrium profile of bank liabilities introduces new effects that we call
money externalities, and which can lead to the formation of competing bank networks of small sizes. It is important to highlight that not only is the discussion of competing networks new in the literature, but also the concept of equilibrium stability that we bring to the discussion, which, in a loose sense, has a tradition to show up in monetary models as questions about robustness, essentiality or tenuousness of equilibria.

The second major modification that we consider is the description of optima under the assumptions of monopoly in inside-money issue, as in the mechanism-design approach of CW. In contrast to the discussing of competition, here we assume that banks cooperate optimally, but we describe, using numerical methods and some proofs, how optimal allocations accommodate aggregate shocks to preferences. We find there main results. (i) For a region of the parameter space, the optimum has banks paying interests on money, instead of contracting the money supply to a level below that of the unrestricted equilibrium (when participation constraints do not bind, which is described by CW for the case of no aggregate uncertainty). (ii) For some parameters, the optimum allocation is asymmetric, in the sense that the response by banks to shocks in periods of loose constraints for nonbanks is different from that in periods of tight constraint for nonbanks. That asymmetry exists even when bank participation constraints do not bind. Loosely speaking, we find that with the mechanism design analysis, the equilibrium shifts some of the distortions of tight-participation periods to the one with weak limitations on incentives. (iii) For the same parameters as (i) and (ii), we find that the optimum has history dependence. That is, after a sequence of constrained periods for nonbank participation, even though a reduction of interests payments is feasible and ex-post desirable, banks reduce the return of money only asymptotically. Like the explanation for (ii), we believe that is also a property of mechanism-design solutions in which distortions in constrained periods are smoothed out to many states to improve ex-ante welfare, although such policies are not ex-post efficient. Results (i), (ii) and (iii), taken together, imply a pattern of returns on money that leave the quantity of money constant, is asymmetric during recessions and expansions, and is path dependent. We believe that our mostly numerical description should add to what is know in the literature about the understanding about optimal policies when there is a fraction of monitored agents in the society, that is, when inside money is essential.
References


Appendix A:

Define \( \theta = (B, \pi, \psi, \beta) \in \Theta = \{[0, 1]^3 \times [0, 1]\} \) and \( \Gamma(\theta) = \{y \in Y_1(\theta) \cap Y_2\} \), where \( Y_1(\theta) = \{y \in \mathbb{R}^{n(n+n_h)} : y \) satisfies restrictions (2) – (11)\} and \( Y_2 = \{y \in \mathbb{R}^{n(n+n_h)} : y_j \leq \tilde{y}_j, \forall j = 1, \ldots, 5n_l \) and \( y_j \leq \tilde{y}_j, \forall j = 5n_l + 1, \ldots, 5(n_l + n_h)\} \).

We are interested in maximizing equation (23), i.e., \( w(y, \theta) \), subject to \( y \in \Gamma(\theta) \). By the Theorem of the Maximum, since \( w \) is continuous and \( \Gamma \) is compact-valued, the function \( h(\theta) = \max_{y \in \Gamma(\theta)} w(y, \theta) \) is well defined and continuous and the correspondence \( G(\theta) = \{y \in \Gamma(\theta) : w(y, \theta) = h(\theta)\} \) of the values that attain the maximum is nonempty, compact-valued and upper hemi-continuous. Moreover, since \( \Gamma \) is also convex-valued and the function \( w \) is strictly concave in \( y \), \( G(\theta) \) is single-valued and, hence, a continuous function.

The optimization problem over the set of implementable, simple, stationary and symmetric inside money allocations has a unique solution that is continuous in \( \theta = (B, \pi, \psi, \beta) \in \Theta = \{[0, 1]^3 \times [0, 1]\} \).

Appendix B:

The solution for the system given by equations (30) and (57)-(59) is:

when \( B \neq 2B_R \):

\[
x^n_0 = \frac{1}{2} \left[ -B - 2B_R + \frac{B^2 + 2B^2}{2B - B_R} - \frac{6B + 4B^2 + 4B_R - 26BB_R + 16B^2}{-2B + 4B_R} + \frac{9B^2 - 6B^3 - 6BB_R + 39B^2 R_R - 24BB_R^2 - 3B^3}{(2B - B_R)(-2B + 4B_R)} \right];
\]

\[
x^n_1 = \frac{1}{2} \left[ 1 + B + B_R - \frac{B^2 - 2B^2}{2B - B_R} + \frac{3B^2 - 2B^3}{2B - B_R} - \frac{13BB_R - 8B^2}{-2B + 4B_R} \right];
\]

\[
x^n_2 = \frac{1}{2} \left[ 1 - 2B + B_R - \frac{3B^2 + 2B^3}{2B - B_R} - \frac{13BB_R + 8B^2}{-2B + 4B_R} \right]
\]

\[
x^G_1 = \frac{3B^2 - 2B^3 + 13BB_R - 8B^2}{2(-2B + 4B_R) - \phi};
\]

\[
x^G_2 = \frac{3B^2 - 2B^3 + 13BB_R - 8B^2}{2(-2B + 4B_R) + \phi};
\]

\[
x^R = B_R
\]

for

\[
\phi = \left[(-3B + 2B^2 + 2B_R - 13BB_R + 8B^2_R)^2 - 4(-2B + 4B_R)(B^2 - 2BB_R + 5B^2 R_R + B^2_R - 8BB_R^2 + 3B^3_R)\right]^\frac{1}{2};
\]

and, when \( B = 2B_R \):

\[
x^n_0 = \frac{1}{2} (1 - 2B_R);
\]

\[
x^n_1 = \frac{1 + 2B_R - 8B^2}{3(2 + 5B_R)};
\]

\[
x^n_2 = \frac{1 - 4B^2}{2(2 + 5B_R)};
\]

\[
x^G = \frac{B_R(1 + 7B_R)}{4 + 10B_R};
\]

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\[ x_G^R = \frac{3BR_1(1+BR_1)}{4+10BR_1}; \]
\[ x_R^R = BR_1. \]

Appendix C:
Define \( \theta = (B, Br, \beta) \in \Theta = \left\{ [0, 1]^2 \times [0, 1] \right\} \) and \( \Gamma (\theta) = \{ y \in Y_1 (\theta) \cap Y_2 \}, \) where \( Y_1 (\theta) = \{ y \in \mathbb{R}_{+}^{19}: y \text{ satisfies restrictions (31) - (56)} \} \) and \( Y_2 = \{ y \in \mathbb{R}_{+}^{19}: y_j \leq \tilde{y}, \forall j = 1, \ldots, 19 \}. \)

Using the same argument showed in appendix A, the optimization problem over the set of implementable, stationary and symmetric inside money allocations implied by the regulation has a unique solution that is continuous in \( \theta. \)

Appendix D:
If constraints do not bind at first-best, we know that all production equals \( y^*. \)

When \( B_R \to 0_+ \), from appendix B, we have that the limit of the distribution of nonbankers and bankers across states is:
\[ x_0^n = x_G^n = \frac{5-2B-\left[4B^2-4B+9\right]}{8}; \]
\[ x_R^n = \frac{1-2B+\left[4B^2-4B+9\right]}{4}; \]
\[ x_G^n = \frac{-3+2B+\left[4B^2-4B+9\right]}{4}; \]
\[ x_R^G = \frac{3+2B-\left[4B^2-4B+9\right]}{4}; \]
\[ x_R^{R} = 0. \]

It is interesting to note that this particular distribution is a mix of red outside money and green inside money banking sector that accepts outside money.

This distribution in addition with equations (68) and (63)-(65) for first-best level of output gives:
\[ RA(B, 0_+) = \Upsilon (B) \frac{\mu (y^*) - y^*}{8S(1 - \beta)B} \]
for \( \Upsilon (B) = -6 - B + (2 + B) \left[4B^2 - 4B + 9\right]^{\frac{1}{2}} - 6B^2. \)

Hence, \( RA(B, 0_+) > 0 \Leftrightarrow \Upsilon (B) > 0 \Leftrightarrow 0 < B < \tilde{B} \simeq 0.12405. \)

When \( B_R \to B_- \), from appendix B, we have that the limit of the distribution of nonbankers and bankers across states is:
\[ x_0^n = x_R^n = \frac{1-B}{2}; \]
\[ x_G^n = x_R^n = x_{\bar{G}}^n = 0; \]
\[ x_R^* = B. \]

It is interesting to note that this particular distribution is a CW one.

This distribution in addition with equations (68) and (63)-(65) for first-best level of output gives\(^{32}\):

\[ RA(B, B_-) = \frac{4B^2u(y^*) - (B + 1)^2 y^*}{2S(1 - \beta)(3B + 1)} \]

Hence, \( RA(B, B_-) < 0 \Leftrightarrow \frac{u(y^*)}{y} < \left( \frac{B + 1}{2B} \right)^2. \)

Summarily,

i) If \( 0 < B < \bar{B} \approx 0.12405 \) and \( \frac{u(y^*)}{y} < \left( \frac{B + 1}{2B} \right)^2 \), then \( RA(B, 0_+) > 0 \) and \( RA(B, B_-) < 0 \).

ii) If \( 0.12405 \approx \bar{B} < B < 1 \) and \( \frac{u(y^*)}{y} > \left( \frac{B + 1}{2B} \right)^2 \), then \( RA(B, 0_+) < 0 \) and \( RA(B, B_-) > 0 \).

**Appendix E:**

When \( B \rightarrow 0_+ \), from appendix B, it is straightforward to see that the limit of the distribution of nonbankers holding red notes is \( x_R^n = \frac{1}{7} \). Suppose that \( 0 \leq x_0^n = x \leq \frac{1}{7} \). Hence, we have \( 0 \leq x_G^n = \frac{1}{2} - x \leq \frac{1}{7} \).

From equation (58), \( \rho^G = \frac{x_G^n}{x_0^n + x_R^n} \rightarrow \frac{1 - 2x}{2(1 - x)} \) and \( \rho_R^n = \frac{x_R^n}{x_0^n + x_R^n} \rightarrow \frac{1}{2(1 - x)}. \)

This distribution in addition with equations (60)-(65) for first-best level of output give:

\[ \psi^R = \frac{1}{S(1 - \beta)} \left\{ \frac{1}{2} (u(y^*) - y^*) \right\}; \tag{71} \]
\[ \psi^G = \frac{1}{S(1 - \beta)} \left\{ xu(y^*) - (1 - x) y^* + \beta \frac{1}{2} \Delta \right\}; \tag{72} \]
\[ \psi_R^G = \frac{1}{S(1 - \beta)} \left\{ \frac{1}{2} u(y^*) - \left( \frac{1}{2} - x \right) y^* - \beta \left( \frac{1}{2} - x \right) \Delta \right\}; \tag{73} \]
\[ \psi_0^n = \frac{1}{S(1 - \beta)} \left\{ -(1 - x) y^* + \beta \frac{1}{2} \Delta_{R0} + \beta \left( \frac{1}{2} - x \right) \Delta_{G0} \right\}; \tag{74} \]
\[ \psi_G^n = \frac{1}{S(1 - \beta)} \left\{ xu(y^*) - \frac{1}{2} y^* - \beta x \Delta_{G0} + \beta \frac{1}{2} \Delta_{RG} \right\}; \tag{75} \]
\[ \psi_R^n = \frac{1}{S(1 - \beta)} \left\{ \frac{1}{2} u(y^*) - \beta x \Delta_{R0} - \beta \left( \frac{1}{2} - x \right) \Delta_{RG} \right\}. \tag{76} \]

\(^{32}\)It can be useful to note that \( \lim_{B \rightarrow 0_+ -} \frac{x_G^n}{x_0^n + x_R^n} = \frac{2B}{2B + 1} \) and \( \lim_{B \rightarrow 0_+ -} \frac{x_R^n}{x_0^n + x_R^n} = \frac{B + 1}{2B + 1} \).
where,

\[
\Delta = v_G^R - v_G^G = \left( \frac{1}{2} - x \right) u(y^*) + \frac{1}{2} y^*; \quad (77)
\]

\[
\Delta_{R0} = v_R^m - v_0^m = \left( \frac{1}{2} u(y^*) + (1 - x) y^* \right); \quad (78)
\]

\[
\Delta_{RG} = v_R^n - v_G^n = \left( \frac{1}{2} - x \right) u(y^*) + \frac{1}{2} y^*; \quad (79)
\]

\[
\Delta_{G0} = v_G^n - v_0^n = \left( \frac{x u(y^*) + (1 - x) y^*}{S(1 - \beta) + \beta} \right); \quad (80)
\]

After some algebra, we have that

\[
v_R = \frac{x v_G^R + x v_G^G}{x v_G^G + x_R} \Leftrightarrow \frac{1}{4} \leq x = \frac{3 + 2 \frac{u(y^*)}{y^*} - \sqrt{5 + 4 \frac{u(y^*)}{y^*}}}{4 \left( 1 + \frac{u(y^*)}{y^*} \right)} \equiv \hat{X}(\frac{u(y^*)}{y^*}) \leq \frac{1}{2}. \quad (81)
\]

It is easy to see from equations (71)-(80) that in a neighborhood of an first-best equilibrium \( v_R^R > v_R^G > v_0^G \), \( v_R > 0, v_G > 0 \) and, when \( \beta \to 1 \), we have that

\[
\frac{u(y^*)}{y^*} > \frac{5}{4} \Rightarrow v_R^G > 0 \quad \text{and} \quad \frac{u(y^*)}{y^*} > 1 \Rightarrow v_0^n > 0 \quad (82)
\]

In order to assure that constraints (31)-(56) do not bind at first-best, we must have \( y^* < \beta (v_R^n - v_G^n) \), \( y^* < \beta v_G^G \), \( y^* < \beta v_R^R \) and \( u(y^*) > \beta (v_R^n - v_0^n) \).

But, when \( \beta \to 1 \),

\[
\left[ x < \frac{1}{2} \left( 1 - \frac{1}{\frac{u(y^*)}{y^*}} \right) \right] \equiv \hat{X}(\frac{u(y^*)}{y^*}) \Rightarrow y^* < \beta (v_R^n - v_G^n) \quad (83)
\]

\[
\frac{u(y^*)}{y^*} > 3 \Rightarrow y^* < \beta (v_G^n - v_R^n) \quad (84)
\]

\[
\frac{u(y^*)}{y^*} > 1 \Rightarrow y^* < \beta v_G^G \quad \text{and} \quad y^* < \beta v_R^R \quad (85)
\]

\[
\frac{u(y^*)}{y^*} > \frac{3}{2} \Rightarrow u(y^*) > \beta (v_R^n - v_0^n) \quad (86)
\]

One can see from (81) and (83) that \( \hat{X}(\cdot) \) cuts \( \hat{X}(\cdot) \) only in \( \frac{u(y^*)}{y^*} = 1 + \sqrt{2} \). Before that point, \( \hat{X}(\cdot) > \hat{X}(\cdot) \) and after that, \( \hat{X}(\cdot) < \hat{X}(\cdot) \).
Hence, if \( u(y^*) > 3 \), the result is reached by a continuity argument.

**Appendix F:**

When \( B \to 1^- \), from appendix B, it is straightforward to see that the limit of the distribution of bankers is \( x^G = x^G_R = \frac{1-B_R}{2} \) and \( x^R = B_R \). From equation (59), \( \rho^G = \frac{x^G}{x^G + x^G_R} \to \frac{1}{2} \) and \( \rho^G_R = \frac{x^G_R}{x^G + x^G_R} \to \frac{1}{2} \).

This distribution in addition with equations (63)-(65) for first-best level of output give:

\[
v^R = \frac{1 + B_R}{S(1 - \beta)} \left\{ \frac{1}{2} (u(y^*) - y^*) \right\} \tag{87} \]
\[
v^G = \frac{1}{S(1 - \beta)} \left\{ (1 - B_R) u(y^*) - y^* + \beta \frac{(1 + B_R)}{2} \Delta \right\} \tag{88} \]
\[
v^G_R = \frac{1}{S(1 - \beta)} \left\{ u(y^*) - (1 - B_R) y^* - \beta \frac{(1 + B_R)}{2} \Delta \right\} \tag{89} \]

where,

\[
\Delta = v^G_R - v^G \tag{90} \]

After some algebra, we have that

\[
v^R = \frac{x^G v^G + x^G_R v^G_R}{x^G + x^G_R} \Rightarrow B_R = \frac{1}{2}. \tag{91} \]

From (60)-(62) and (87)-(90), when \( B \to 1^- \) and \( B_R = \frac{1}{2} \):
But, when $eta \rightarrow 1$, we can derive that

$$\frac{u(y^*)}{y^*} > 1 \Rightarrow y^* < \beta (v_R^n - v_G^n) \quad (103)$$

$$\frac{u(y^*)}{y^*} > \frac{27}{13} \Rightarrow y^* < \beta (v_G^n - v_R^n) \quad (104)$$

$$\frac{u(y^*)}{y^*} > 1 \Rightarrow y^* < \beta v_R^G \text{ and } y^* < \beta v_R^R \quad (105)$$

$$\frac{u(y^*)}{y^*} > \frac{27}{17} \Rightarrow u(y^*) > \beta (v_R^n - v_G^n) \quad (106)$$
Hence, if \( \frac{u(y)}{y} > 3 \), the result is reached by a continuity argument.

**Appendix G: Case with green banks not accepting red notes**

Suppose that green banks do not accept red notes and take by hypothesis that allocation derived from this new regulation when all production equals \( y^* \) is implementable.

In this case, we would have the following distributions across states

\[
x^R = B_R, \quad x^G = B - B_R \quad \text{and} \quad x^0_0 + x^0_G + x^R_R = (1 - B)
\]  

(107)

for \( 0 \leq B_R \leq B \leq 1 \).

The stationarity across states implies that the inflow into state 0 and \( G \) must equal the outflow from state 0 and \( G \), respectively, for nonbankers, i.e.:

\[
x^0_G [B - B_R] + x^0_R B_R = x^0_0 B
\]  

(108)

\[
x^0_0 [B - B_R] = x^0_G B
\]  

(109)

Equations (107) up to (109) supply a system of linear equations whose the solution is:

\[
x^0_0 = \frac{(1 - B) B}{2(2B - B_R)};
\]  

(110)

\[
x^0_G = \frac{(1 - B)(B - B_R)}{2(2B - B_R)};
\]  

(111)

\[
x^0_R = \frac{1 - B}{2};
\]  

(112)

\[
x^G = B - B_R;
\]  

(113)

\[
x^R = B_R.
\]  

(114)

A competitive equilibrium can also be found when \( RA(B, B_R) = 0 \), but where

\[
RA(B, B_R) \equiv v^R - v^G
\]  

(115)
Proposition 8 Suppose that constraints do not bind at first-best:

i) If $B \in (0, 1)$ is sufficiently small, i.e., $0 < B < 0.2$, and $\frac{u(y^*)}{y} < \left( \frac{B+1}{2B} \right)$, then $\lim_{B \to 0} RA(B, B_R) > 0$ and $\lim_{B \to B_R} RA(B, B_R) < 0$.

ii) If $B \in (0, 1)$ is sufficiently large, i.e., $0.2 < B < 1$, and $\frac{u(y^*)}{y} > \left( \frac{B+1}{2B} \right)$, then $\lim_{B \to 0} RA(B, B_R) < 0$ and $\lim_{B \to B_R} RA(B, B_R) > 0$.

**Proof.** If constraints do not bind at first-best, we know that all production equals $y^*$.

When $B_R \to 0$, from (110) up to (114), we have that the limit of the distribution of nonbankers and bankers across states is:

$$
x_0^n = x_G^n = \frac{(1-B)}{4};
$$

$$
x_R^n = \frac{(1-B)}{2};
$$

$$
x_G^G = B;
$$

$$
x_R^R = 0.
$$

This distribution in addition with equation (115) and the value functions for first-best level of output gives:

$$
RA(B, 0_+) = \frac{(1 - 5B) [u(y^*) - y^*]}{4S(1 - \beta)B}.
$$

Hence, $RA(B, 0_+) > 0 \iff 0 < B < 0.20$.

When $B_R \to B$, from (110) up to (114), we have that the limit of the distribution of nonbankers and bankers across states is:

$$
x_0^n = x_G^n = \frac{1-B}{2};
$$

$$
x_R^n = x_G^G = x_R^G = 0;
$$

$$
x_R^R = B.
$$

This distribution in addition with equation (115) and the value functions for first-best level of output gives:

$$
RA(B, B_-) = \frac{[2Bu(y^*) - (B + 1)y^*]}{2S(1 - \beta)}
$$

Hence, $RA(B, B_-) < 0 \iff \frac{u(y^*)}{y^*} < \left( \frac{B+1}{2B} \right)$.

Summarily,
i) If $0 < B < 0.2$ and $\frac{u(y^*)}{y^*} < \left( \frac{B+1}{2B} \right)$, then $RA(B, 0_+) > 0$ and $RA(B, B_-) < 0$.

ii) If $0.2 < B < 1$ and $\frac{u(y^*)}{y^*} > \left( \frac{B+1}{2B} \right)$, then $RA(B, 0_+) < 0$ and $RA(B, B_-) > 0$. ■

**Proposition 9** If $B \in (0, 1)$ is sufficiently small and $\beta \in (0, 1)$ and $\frac{u(y^*)}{y^*}$ are sufficiently large, then the first-best equilibrium is not implementable.

**Proof.** When $B \to 0$, from (110) up to (114), it is straightforward to see that the limit of the distribution of nonbankers holding red notes is $x^R_n = \frac{1}{2}$. Suppose that $0 \leq x^R_0 = x \leq \frac{1}{2}$. Hence, we have $0 \leq x^R_0 = \frac{1}{2} - x \leq \frac{1}{2}$.

This distribution in addition with the value functions\(^{33}\) for first-best level of output give:

\[
\begin{align*}
v^R &= \frac{1}{S(1 - \beta)} \left\{ \frac{1}{2} (u(y^*) - y^*) \right\} ; \quad (116) \\
v^G &= \frac{1}{S(1 - \beta)} \left\{ xu(y^*) - \left( \frac{1}{2} - x \right) y^* \right\} ; \quad (117) \\
v^n_0 &= \frac{1}{S(1 - \beta)} \left\{ -(1 - x) y^* + \beta \frac{1}{2} \Delta_R + \beta \left( \frac{1}{2} - x \right) \Delta_G \right\} ; \quad (118) \\
v^n_G &= \frac{1}{S(1 - \beta)} \left\{ xu(y^*) - \frac{1}{2} y^* - \beta x \Delta_G + \beta \frac{1}{2} \Delta_{RG} \right\} ; \quad (119) \\
v^n_R &= \frac{1}{S(1 - \beta)} \left\{ \frac{1}{2} u(y^*) - \beta x \Delta_R - \beta \left( \frac{1}{2} - x \right) \Delta_{RG} \right\} ; \quad (120)
\end{align*}
\]

where,

\[
\begin{align*}
\Delta_R &= v^n_R - v^n_0 = \frac{\left\{ \frac{1}{2} u(y^*) + (1 - x) y^* \right\}}{S(1 - \beta) + \beta} ; \quad (121) \\
\Delta_{RG} &= v^n_R - v^n_G = \frac{\left\{ \left( \frac{1}{2} - x \right) u(y^*) + \frac{1}{2} y^* \right\}}{S(1 - \beta) + \beta} ; \quad (122) \\
\Delta_G &= v^n_G - v^n_0 = \frac{\left\{ xu(y^*) + \left( \frac{1}{2} - x \right) y^* \right\}}{S(1 - \beta) + \beta} . \quad (123)
\end{align*}
\]

After some algebra, we have that

\[
RA(B, B_R) = 0 \iff v^R = v^G \iff \frac{1}{4} \leq x = \frac{1}{2 \left( \frac{1}{1 + \frac{1}{2B}} \right)} \equiv \hat{X} \left( \frac{u(y^*)}{y^*} \right) \leq \frac{1}{2}. \quad (124)
\]

\(^{33}\)Here we omitted the value functions, but one can write them easily in the same fashion as (60)-(65).
It is easy to see from equations (116) up to (123) that in a neighborhood of an first-best equilibrium
\( v^n_R > v^n_G > v^n_0 \), \( v^R = v^G > 0 \) and, when \( \beta \to 1 \), we have that
\[
\frac{u(y^*)}{y^*} > 1 \Rightarrow v^n_0 > 0
\]  

(125)

In order to assure that constraints\(^{34}\) do not bind at first-best, we must have \( y^* < \beta (v^n_R - v^n_G) \), \( y^* < \beta v^G \), \( y^* < \beta v^R \) and \( u(y^*) > \beta (v^n_R - v^n_0) \).

But, when \( \beta \to 1 \),
\[
\left[ x < \frac{1}{2} \left( 1 - \frac{1}{u(y^*)} \right) \equiv \bar{X} \left( \frac{u(y^*)}{y^*} \right) \right] \Rightarrow y^* < \beta (v^n_R - v^n_G) \]  

(126)

\[
\frac{u(y^*)}{y^*} > 3 \Rightarrow y^* < \beta (v^n_R - v^n_G) \]  

(127)

\[
\frac{u(y^*)}{y^*} > 1 \Rightarrow y^* < \beta v^G = \beta v^R \]  

(128)

\[
\frac{u(y^*)}{y^*} > \sqrt{2} \Rightarrow u(y^*) > \beta (v^n_R - v^n_0) \]  

(129)

One can see from (124) and (126) that \( \bar{X} \left( \frac{u(y^*)}{y^*} \right) > \bar{X} \left( \frac{u(y^*)}{y^*} \right) \), \( \forall \frac{u(y^*)}{y^*} \) such as \( 1 \leq \frac{u(y^*)}{y^*} < \infty \).

Hence, even if \( \beta \to 1 \), this competitive equilibrium is not implementable, since nonbankers holding green notes would never produce to receive a red note. ■

**Proposition 10** If \( B \in (0, 1) \), \( \beta \in (0, 1) \) and \( \frac{u(y^*)}{y^*} \) are sufficiently large, then the first-best equilibrium is not implementable.

**Proof.** When \( B \to 1 \), from (110) up to (114), it is straightforward to see that the limit of the distribution of bankers is \( x^G = 1 - B_R \) and \( x^R = B_R \).

This distribution in addition with the value functions for first-best level of output give:

\[
v^R = \frac{B_R}{S(1 - \beta)} \left\{ u(y^*) - y^* \right\} ; \quad (130)
\]

\[
v^G = \frac{1 - B_R}{S(1 - \beta)} \left\{ u(y^*) - y^* \right\} ; \quad (131)
\]

\(^{34}\)Here we omitted the constraints, but one can write them easily in the same fashion as (31)-(56).
After some algebra, we have that

$$v^R = v^G \iff B_R = \frac{1}{2}$$

(132)

From the value function and equations (130) and (131), when $B \to 1$ and $B_R = \frac{1}{2}$:

$$
\begin{align*}
v^R & = v^G = \frac{1}{2S (1 - \beta)} \{u(y^*) - y^*\}; \\
v^0_0 & = -8S^2 (1 - \beta)^2 y^* + S (1 - \beta) \beta [4u(y^*) - 14y^*] + \beta^2 [4u(y^*) - 5y^*] \\
v^0_0 & = \frac{4S^2 (1 - \beta)^2 [u(y^*) - y^*] + S (1 - \beta) \beta [8u(y^*) - 10y^*] + \beta^2 [4u(y^*) - 5y^*]}{4S (1 - \beta) \beta [2S (1 - \beta) + 3\beta]} \\
v^0_0 & = \frac{4S^2 (1 - \beta)^2 u(y^*) + S (1 - \beta) \beta [8u(y^*) - 4y^*] + \beta^2 [4u(y^*) - 5y^*]}{4S (1 - \beta) \beta [2S (1 - \beta) + 3\beta]} \\
\Delta G & = v^0_0 - v^0_0 = \frac{[u(y^*) + y^*]}{2S (1 - \beta) + 3\beta}; \\
\Delta RG & = v^0_0 - v^0_0 = \frac{y^* [S (1 - \beta) + \frac{3\beta}{2}]}{[S (1 - \beta) + \beta \beta [2S (1 - \beta) + 3\beta]]}; \\
\Delta R0 & = v^0_0 - v^0_0 = \frac{u (y^*) [S (1 - \beta) + \beta + y^* [2S (1 - \beta) + \frac{5\beta}{2}]}{[S (1 - \beta) + \beta [2S (1 - \beta) + 3\beta]].}
\end{align*}
$$

(133) to (139)

It is easy to see from equations (133) up to (139) that in a first-best equilibrium $v^0_0 > v^0_0 > v^0_0$, $v^R = v^G > 0$ and, when $\beta \to 1$, we have that

$$
\frac{u(y^*)}{y^*} > \frac{5}{4} \Rightarrow v^0_0 > 0
$$

(140)

In order to assure that constraints do not bind at first-best, we must have $y^* < \beta (v^0_0 - v^0_0)$, $y^* < \beta (v^0_0 - v^0_0)$, $y^* < \beta v^G$, $y^* < \beta v^R$ and $u (y^*) > \beta (v^0_0 - v^0_0)$.

When $\beta \to 1$,

$$
\begin{align*}
\frac{u(y^*)}{y^*} & > 2 \Rightarrow y^* < \beta (v^0_0 - v^0_0) \\
\frac{u(y^*)}{y^*} & > 1 \Rightarrow y^* < \beta v^G = \beta v^R \\
\frac{u(y^*)}{y^*} & > \frac{5}{4} \Rightarrow u (y^*) > \beta (v^0_0 - v^0_0)
\end{align*}
$$

(141) to (143)

However, we can not assure, even when $\beta \to 1$, that $y^* < \beta (v^0_0 - v^0_0)$. 

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Hence, even if $\beta \to 1$, this competitive equilibrium is not implementable, since nonbankers holding green notes would never produce to receive a red note. ■