Approximate Recursive Equilibrium with Minimal State Space

Raad, Rodrigo Jardim

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Approximate Recursive Equilibrium with Minimal State Space

Raad, R.

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Departament of Economics, Federal University of Minas Gerais, UFMG - Brazil.

Abstract

This paper shows existence of approximate recursive equilibrium with minimal state space in an environment of incomplete markets. We prove that the approximate recursive equilibrium implements an approximate sequential equilibrium which is always close to a Magill and Quinzii equilibrium without short sales for arbitrarily small errors. This implies that the competitive equilibrium can be implemented by using forecast statistics with minimal state space provided that agents will reduce errors in their estimates in the long run. We have also developed an alternative algorithm to compute the approximate recursive equilibrium with incomplete markets and heterogeneous agents through a procedure of iterating functional equations and without using the first order conditions of optimality.

Keywords: Recursive Equilibrium, Computational Economics, Incomplete Markets, Approximate Equilibrium.

JEL Classification: D50, D52.

1 Introduction

Magill and Quinzii (1996) provide an extension of the representative agent models such as Lucas Jr (1978) to an economy with infinite lived heterogeneous agents in which markets can be incomplete. They analyzed the
equilibrium properties in sequential markets under the assumption of common expectations on prices as in Radner (1972). In this model, markets are incomplete in the sense that at every date and for every commodity there will be some future dates and some events at those dates for which it will not be possible to make current contracts for future delivery contingent on those events. In this way, agents make consumption and investment plans contingent on the realizations of possible states of nature, under their own beliefs, basing the current choices on the optimal planning. The latter depends on how the agents have expectations on prices contingent on the fulfillment of future events representing exogenous uncertainty, i.e., events whose realizations are independent of agents’ choices. This fact is the essence of the common expectations hypothesis.

There are two natural ways to explain how this equilibrium with common expectations can be achieved. The first regards to the high level of coordination among agents who propose to only negotiate at that contingent price level matched in the future. The second takes into account the computing capacity of the agents who use past variables and economic fundamentals to obtain a recursive relation between endogenous variables in two consecutive periods characterized, basically, by recursive transition functions. This set of functions, also called recursive equilibrium,\(^1\) includes price estimators that represent a common recursive price expectation and implements the equilibrium in the sequential markets. Duffie et al. (1994) has a fairly general result showing the existence of such estimators but with a domain containing a large number of variables, which calls into question the assumption that agents can compute the prices accurately. Kubler and Polemarchakis (2004), Spear (1985) and Hellwig (1982) point to a possible generic nonexistence of recursive equilibrium, with few variables in its domain, for the model of overlapping generations. But Kubler and Polemarchakis (2004) show that it is possible to obtain a stream of prices implemented recursively where agents trade goods and assets that makes its optimal utility quite close to that obtained if prices actually confirm their expectations.

The methodology used in the literature to construct a recursive equilibrium is given in Duffie et al. (1994). Basically, they consider a state space \(S\) containing all pay-off relevant variables and a correspondence \(G : S \rightarrow\)

\(^1\)Sometimes we find in the literature references to recursive equilibrium as the sequential equilibrium implemented by the transition functions. Here, we find it more convenient to call the recursive equilibrium a set of transition functions, since we show that these functions implement the sequential equilibrium.
Prob($S$), where Prob($S$) is the set of Borel probability measures over $S$. This correspondence is interpreted as the intertemporal consistency, derived from some particular model, embodying exogenous shocks, feasibility conditions, and the first order optimality conditions of dynamic programming problems of the agents. A measurable subset $S' \subset S$ is said to be self-justified if $G(s) \cap \text{Prob}(S') \neq \emptyset$ for all $s \in S'$. Under regular assumptions on $G$, DGMM show that, if there exists a non-empty compact self-justified set $S' \subset S$, then $G$ admits a measurable selection and, using the Skorokhod’s Theorem, they find a stationary function $g$ defined in $S'$ relating two consecutive realizations of the equilibrium stochastic process. The approximate recursive equilibrium defined in Kubler and Polemarchakis (2004) is a function whose domain contains few variables and that approximates the selector $g$, but might be very far from exact equilibrium as presented in Kubler and Schmedders (2005). The latter paper exhibits an example that illustrates this fact, with zero intertemporal discount rate and the estimation error consisting only in the first order conditions in the optimization problem of the agents. Moreover, they develop theoretical foundations for an error analysis of approximate equilibrium providing sufficient conditions to ensure that approximate equilibrium are close to exact equilibrium. Kubler (2011) also examines the relationship between exact and approximate recursive equilibrium for economies whose fundamentals are semi-algebraic, that is, instantaneous utility and production functions can be described by finitely many polynomials.

In this paper, we show the possibility of a sequential equilibrium be eventually achieved (and computed) with common expectations as in Radner (1972) or Magill and Quinzii (1996), even without assuming a high level of coordination among agents. As in Duffie et al. (1994) and in Kubler and Polemarchakis (2004), we construct estimators that implement the prices and allocations of consumption and asset shares so that agents maximize their expected utility but with arbitrarily small errors in the market clearing conditions. However, these estimators are constructed through a selection of a correspondence defined without using the first order conditions of optimality and so that one can always implement a sequential equilibrium arbitrary closed to an exact equilibrium. We do not need to assume further conditions.

\footnote{This function also depends on an extra coordinate that represents the effect of an uniform exogenous shock on the equilibrium. In some models it is possible to prove that actually, the recursive function does not depend on this process and, in this case, is called spotless recursive equilibrium.}
such as in Kubler (2011) and, contrary to the definition of approximate equilibrium given in Kubler and Polemarchakis (2004), the optimality choices are exact in our approach, avoiding the criticism of Kubler and Schmedders (2005). We assume that the demand for current assets can not expand arbitrarily as a function of asset endowments in the previous period and current prices, constructing an example with one good and one asset clarifying this assumption. Furthermore, we show that these estimators are continuous and contains a domain with the smallest possible number of variables, also called minimal state space.\(^3\) Contrary to Duffie et al. (1994), we use a constructive argument explaining how, in sequential markets, the equilibrium can be implemented recursively by the transition functions by showing explicitly the consecutive relations among endogenous variables. Finally, we show an alternative way to compute an equilibrium, with heterogeneous agents obtaining the estimators, without using the first order conditions, through a procedure of iterating functional equations. Computing the value function and the demand through its argmax, we obtain approximations whose errors do not propagate over time as pointed out by Kubler and Schmedders (2005) about the method of Scarf (1967) which approximates the first order conditions. Indeed, the value function is the optimal value among the current choices and all feasible plans over all future periods and the errors are generated only through the value function.

2 The model

2.1 Definitions

Suppose that there exist a finite set of types denoted by \(I = \{1, \ldots, I\}\) and such that each type \(i \in I\) has a continuum of agents trading in a competitive environment. Time is indexed by \(t\) in the set \(\mathbb{N} = \{1, 2, \ldots\}\) for current periods and \(r \in \mathbb{N} \cup \{0\}\) for future periods. In this model, the uncertainty is exogenous, in the sense of being independent of agents’ actions. Each agent knows the whole set of possible exogenous variables, also called states of the nature, and trade contingent claims. Let \(Z\) be a finite set containing all states of the nature and \(\mathcal{Z}\) its \(\sigma\)-algebra. Denote by \((\mathcal{Z}_t, \mathcal{Z}_t')\) a copy of

\(^3\)This set contains the portfolio asset endowment of previous period and the current state of nature. This set is minimal because an asset redistribution must change the equilibrium prices if, for instance, there exists agents with heterogeneity in the risk aversion.
\((Z, \mathcal{Z})\) for all \(t \in \mathbb{N}\). Exogenous uncertainty is described by the streams \(z^t = (z_1, ..., z_t) \in Z_1 \times \cdots \times Z_t = Z^t\) for all \(t \in \mathbb{N}\), that is, the set of nodes of the event tree is given by \(\bigcup_{t \in \mathbb{N}} Z^t\).

There are a finite set \(\mathcal{J} = \{1, ..., J\}\) of goods and a finite set \(\mathcal{H} = \{1, ..., H\}\) of long lived real assets in net supply equal to one and with dividends characterized by measurable bounded functions \(\hat{d} : Z \to \mathbb{R}^{+H}\) in units of goods. The matrix element \(\hat{d}_{jh}(z)\) of \(\hat{d}(z)\) represents the amount of good \(j\) paid by one unit of asset \(h\) in the state of the nature \(z\). Denote by \(\Theta^i \subset \mathbb{R}^H_+\) for all \(i \in \mathcal{I}\) the convex set where asset choices are defined and \(C^i \subset \mathbb{R}^J_+\) the convex set where agent \(i\)'s consumption is chosen. Observe that we are not allowing for short-sales. Define the symbol without upper index as the Cartesian product. For instance, write \(C = \prod_{i \in \mathcal{I}} C^i\).

Denote by \(Q = \{(q^i, q^a) \in \mathbb{R}^J_+ \times \mathbb{R}^H_+ : \sum_{j \in \mathcal{J}} q^i_j + \sum_{h \in \mathcal{H}} q^a_h = 1\}\) the set where the prices are defined and write \(Q^o = Q \cap \mathbb{R}^{J+H}_+\). The symbol \(q = (q^i, q^a) \in Q\) stands for the consumption and asset prices in units of the numéraire respectively. We write \(q^i \hat{d}_h(z) := \sum_{j \in \mathcal{J}} q^i_j \hat{d}_{jh}(z) \in \mathbb{R}_+\) as the amount of the numéraire paid by one unit of asset \(h\) and write \(q^a \hat{d}(z)\theta^i := \sum_{h \in \mathcal{H}} q^a_h \hat{d}_h(z)\theta^i_h \in \mathbb{R}_+\) as the amount of the numéraire paid by the portfolio \(\theta^i \in \mathbb{R}^H_+\). In addition, we define \(\hat{d}(z)\theta^i = \sum_{h \in \mathcal{H}} \hat{d}_h(z)\theta^i_h \in \mathbb{R}^J_+\) for all \(\theta^i \in \Theta^i\) as the amount of goods 1, ..., \(J\) resulting from the ownership of the portfolio \(\theta^i\).

Write \(\Theta = \{\hat{\theta} \in \Theta : \sum_{i \in \mathcal{I}} \hat{\theta}^i_h = 1\) for all \(h \in \mathcal{H}\}\). An element \(\hat{\theta} \in \Theta\) stands for the mean aggregate asset choice of the agents.

Let \(S = \Theta \times Z\) be the space of state variables endowed with the product topology with a typical element denoted by \(s = (\hat{\theta}, z)\). Write \(\mathcal{I}\) the Borel subsets of \(S\) and \((S_t, \mathcal{I}_t)\) a copy of \((S, \mathcal{I})\) for all \(t \in \mathbb{N}\). The set \(S_t\) contains the variables on which the beliefs will be defined. Write the set of all continuous\(^4\) functions \(\hat{q} : S \to Q\) by \(\hat{Q}\) and the set of all continuous functions \(\hat{q} : S \to Q^o\) by \(\hat{Q}^o\).

Every Cartesian product of topological spaces is endowed with the product topology and any set of bounded continuous functions is endowed with the topology induced by the sup norm. The norm \(\| \cdot \|\) in \(\mathbb{R}^n\) considered here is the max norm, that is, \(\|x\| = \max\{|x_1|, ..., |x_n|\}\).

The instantaneous utility is a bounded real valued function \(u^i : \mathbb{R}^J_+ \to \mathbb{R}_+\) strictly concave and strictly increasing for all \(i \in \mathcal{I}\) where the symbol \(\partial_j u^i(e^t)\)

\(^4\)And hence bounded because \(S\) is compact.
stands for the partial derivative of \( u^i \) with respect to the \( j \)-th coordinate evaluated at the point \( c^i \).

Each agent \( i \) has a measurable endowment \( e^i : Z \to \mathbb{R}_+^J \) of goods 1, ..., \( J \).

### 2.2 Agents’ features

Agents’ beliefs at every fixed date \( r \) are characterized by the continuous\(^5\) map \( \mu^i_r : Z \to \text{Prob}(Z^r) \) for \( r \in \mathbb{N} \), anticipating future exogenous states of the nature given the realization of the current state of nature \( z \). We suppose that these beliefs are \textit{predictive} in the context of Blackwell and Dubins (1962) with continuous probability transition rules \( \lambda^i : Z \to \text{Prob}(Z) \) where \( \text{Prob}(Z) \) is endowed with the weak topology. More precisely, the measure \( \mu^i_r \) satisfies\(^6\)

\[
\mu^i_r(z)(A_1, ..., A_r) = \int_{A_1} \cdots \int_{A_r} \lambda^i(z_{r-1}, dz_r) \cdots \lambda^i(z, dz_1).
\]

\textit{Remark 2.3.} Consider a stochastic process \( \{\tilde{z}_t\}_{t \in \mathbb{N}} \) defined on some underlying probability space and with range in the probability space \( Z \). A probability transition rule on the period \( t \) can be regarded as a generalization of the notion of conditional probability of \( \tilde{z}_{t+1} \) given \( \tilde{z}_t \) with respect to agents’ subjective probability within the framework of Savage (1954).

We follow the approach of contingent choices as given in Radner (1972). As the model described here does not assume market completeness, then agents may not be unwilling to make contracts that implement some future path of consumption as in the Arrow-Debreu equilibrium. Since agents do not perfectly anticipate the future states of nature, which are given exogenously, rationality leads them to make plans for the future at each current period contingent to all possible future trajectories of the states of nature. Therefore we have the definition below.

\textbf{Definition 2.4.} The agent \( i \)'s plan at some period is defined as the current period choice \( (c^i_0, \theta^i_0) \in C^i \times \Theta^i \) and the streams \( \{c^i_r\}_{r \in \mathbb{N}} \) and \( \{\theta^i_r\}_{r \in \mathbb{N}} \) of measurable functions \( c^i_r : Z^r \to C^i \) and \( \theta^i_r : Z^r \to \Theta^i \) for all \( r \in \mathbb{N} \) representing future plans.

\(^5\)The set \( \text{Prob}(Z^r) \) is endowed with the weak topology.

\(^6\)See Stokey et al. (1989) chapters 8 and 9 for details about the topology of \( \text{Prob}(Z) \), the construction of a probability measure based on the composition of probability transition rules and results about expectations over this measure.
Remark 2.5. In each current period, the quantity $c_i^r(z^r)$ can be interpreted as the value planned for consumption $r$ periods ahead if $z^r$ is the partial history of prices actually observed at these periods. The asset plan $\{\theta_i^r\}_{r \in \mathbb{N}}$ has analogous interpretation.

We assume that the agents choose a feasible plan of consumption and investment that maximizes the expected utility, under their own beliefs, among all other feasible plans. The next definitions characterize the feasibility of a plan and how agents calculate its expected value.

**Definition 2.6.** Let $B^i : \Theta^i \times Z \times Q \to C^i \times \Theta^i$ be defined as

$$B^i(\theta^i, z, q) = \{(c^i, \theta^i) \in C^i \times \Theta^i : q^e c^i + q^a \theta^i \leq (q^a + q^c \hat{d}(z)) \theta^i + q^c e^i(z)\}.$$  

Let $q = \{q_r : Z^r \to Q\}_{r \geq 0}$ be a stream of contingent prices for a given $q_0 \in Q$. A plan $(c^i, \theta^i)$, for an agent $i \in \mathcal{I}$, is feasible from $(\theta^i, z, q)$ if $(c^i_0, \theta^i_0) \in B^i(\theta^i, z, q_0)$ and

$$(c^i_r(z^r), \theta^i_r(z^r)) \in B^i(\theta^i_{r-1}(z^{r-1}), z_r, q_r(z^r)) \text{ for all } z^r \in Z^r$$

and for all $r \in \mathbb{N}$.

Denote by $\mathcal{F}^i(\theta^i, z, q)$ the set of all feasible plans from $(\theta^i, z, q)$.

Now we can define the expected utility.

**Definition 2.7.** Let $C^i$ be the set of all sequence of measurable functions $\{c^i_r : Z^r \to C^i\}_{r \geq 0}$ with $c^i_0 \in C^i$ for $r \in \mathbb{N}$. We define the agent $i$’s expected utility $U^i : C^i \times Z \to \mathbb{R}$ for consuming $c^i$ given the state $z \in Z$:

$$U^i(c^i, z) = u^i(c^i_0) + \sum_{r \in \mathbb{N}} \int_{Z^r} \beta^r u^i(c^i_r(z^r)) \mu^i_r(z, dz^r).$$

**Definition 2.8.** Define the value function $\hat{v}^i : \Theta^i \times Z \to \mathbb{R}$ by:

$$\hat{v}^i(\theta^i, z, q) = \sup \{U^i(c^i, z) : (c^i, \theta^i) \in \mathcal{F}^i(\theta^i, z, q)\}. \quad (1)$$

The following definition characterizes the agents’ demand. Although this demand is independent of time, it yields the current choice at period $t$ given some past and current observed variables for each $t \in \mathbb{N}$. This approach allows us to use the recursive methods and compute the sequential equilibrium.
Definition 2.9. We define the agent $i$’s demand for good and asset by:

$$
\delta^i(\theta^i, z, q) = \arg\max \left\{ U^i(c^i, z) : (c^i, \theta^i) \in F^i(\theta^i, z, q) \right\}.
$$

The result below can be found in Raad (2012) using that the intertemporal budget correspondence is continuous in the product topology and the Berge Maximum Theorem.

Proposition 2.10. The value function $\hat{v}^i$ is continuous on the set

$$
\{(\theta^i, z, q) : (q_0^i + q_0^i \hat{d}(z))\theta^i + q_0^i c^i(z) > 0\}.
$$

The lemma below ensures that the demand becomes arbitrarily large for arbitrarily low prices.

Lemma 2.11. For each compact $\tilde{C} \times \tilde{\Theta} \subset \text{Int}(C \times \Theta)$ there exists $\gamma > 0$ such that if $(\tilde{\theta}, z, c, \tilde{\theta}, q)$ satisfies $(c^i, \theta^i) \in \delta^i(\tilde{\theta}^i, z, q)$, $(c_0^i, \theta_0^i) \in \tilde{C}^i \times \tilde{\Theta}^i$ and $(c^i_r(z^r), \theta^i_r(z^r)) \in \tilde{C}^i \times \tilde{\Theta}^i$ for all $z^r \in Z^r$ and all $i \in I$ then $q_0 \geq \gamma$.

Proof: See Lemma 7.10 in the Appendix. \qed

3 Recursive and sequential equilibrium

Roughly speaking, we can say that a recursive equilibrium is a function relating the variables of equilibrium between two consecutive periods. Duffie et al. (1994) shows the existence of recursive equilibrium in models with heterogeneous agents and state space containing all pay-off relevant variables. Kubler and Polemarchakis (2004) argues that a recursive equilibrium having a reduced state space may not exist in a model of overlapping generations and heterogeneous agents. Furthermore, Kubler and Schmedders (2002) also shows an example of the non existence of recursive equilibrium for the case of a minimal space state pointing to the study of approximate recursive equilibrium as a solution to the problem of coordination and the calculation of the sequential equilibrium numerically. This is our goal hereafter.

The next definition specifies the sequential equilibrium of the economy, also called competitive equilibrium.

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7 This correspondence can be empty when $C^i \times \Theta^i$ is not compact.

8 Recall that the state space is defined as the domain of the recursive functions.
Definition 3.1. Consider the following measurable

1. \( q := \{ q_r : Z^r \to Q^o \}_{r \geq 0} \) contingent prices;
2. \( c := \{ c_r : Z^r \to C \}_{r \geq 0} \) contingent consumption allocation;
3. \( \theta := \{ \theta_r : Z^r \to \Theta \}_{r \geq 0} \) contingent portfolio allocation.

Let \( \bar{\theta} \) be a previous portfolio distribution and \( z \) a current state of the nature in a period \( t \). Then these allocations and prices constitute an equilibrium for \( E \) in a period \( t \) if satisfy for all \( z^r \in Z^r \):

1. optimality: \((c^i, \theta^i) \in \delta^i(\bar{\theta}^i, z, q)\);
2. asset markets clearing: \( \sum_{i \in I} \theta^i_r(z^r) = 1 \in \mathbb{R}^H \);
3. good markets clearing: \( \sum_{i \in I} c^i_r(z^r) = d(z_r) \cdot 1 + \sum_{i \in I} e^i(z_r) \).

The concept approximate sequential equilibrium is similar to that given in Kubler and Polemarchakis (2004), consisting of a set of allocations of consumption and portfolio and a stream of prices balancing all markets approximately. However, contrary to Kubler and Polemarchakis (2004), here agents maximize the expected utility achieving the exact optimal benefit at that level of prices.

Definition 3.2. Given \( \epsilon > 0 \), consider the following measurable

1. \( q := \{ q_r : Z^r \to Q^o \}_{r \geq 0} \) contingent prices;
2. \( c := \{ c_r : Z^r \to C \}_{r \geq 0} \) contingent consumption allocation;
3. \( \theta := \{ \theta_r : Z^r \to \Theta \}_{r \geq 0} \) contingent portfolio allocation.

Let \( \bar{\theta} \) be a previous portfolio distribution and \( z \) a current state of the nature. Then these allocations and prices constitute an \( \epsilon \)-approximate sequential equilibrium for \( E \) in some period if satisfy for all \( z^r \in Z^r \):

1. \((c^i, \theta^i) \in \delta^i(\bar{\theta}^i, z, q)\);
2. \( \| \sum_{i \in I} \theta^i_r(z^r) - 1 \| \leq \epsilon \);
3. \( \| \sum_{i \in I} c^i_r(z^r) - e^i(z_r) - d(z_r) \cdot 1 \| \leq \epsilon \).

\(^9\)Define \( 1 \) as the vector \((1, ..., 1) \in \mathbb{R}^H \).
We introduce now the concept of approximate recursive equilibrium and show in the appendix that it implements the sequential equilibrium of the economy. The recursive demand will be constructed using the value function. The latter is defined as the optimal value among all feasible plans, given the income and current portfolio endowments and, in addition, the transitions of the endogenous variables such as prices and asset distribution.

Write \( \hat{\Theta} \) the space of all continuous functions \( \hat{\theta} : S \rightarrow \Theta \) representing the transition of asset distributions. A well known result states that for each \( i \in I \) there exists a bounded value function \( v^i : \Theta^i \times S \times \hat{Q}^o \times \hat{\Theta} \rightarrow \mathbb{R} \) satisfying the Bellman Equation:

\[
v^i(\hat{\theta}^i, s, \hat{q}, \hat{\theta}) = \sup \left\{ u(c^i) + \beta \int_z v^i(\hat{\theta}^i(s), \hat{q}, \hat{\theta}) \lambda^i(z, dz') \right\}
\]

over all \( (c^i, \hat{\theta}^i) \in B^i(\hat{\theta}^i, z, \hat{q}(s)) \). Indeed, write \( \mathbb{V} \) the space of all bounded value functions \( v^i : \Theta^i \times S \times \hat{Q}^o \times \hat{\Theta} \rightarrow \mathbb{R} \) endowed with the sup norm and consider the operator \( T^i : \mathbb{V} \rightarrow \mathbb{V} \), defined by

\[
T^i(v^i)(\theta^i, s, \hat{q}, \hat{\theta}) = \sup \left\{ u(c^i) + \beta \int_z v^i(\theta^i(s), \hat{q}, \hat{\theta}) \lambda^i(z, dz') \right\}
\]

over all \( (c^i, \theta^i) \in B^i(\theta^i, z, \hat{q}(s)) \). Clearly, \( T \) satisfies Blackwell’s sufficient conditions for a contraction and hence has a fixed point. See Stokey et al. (1989) for further details.

**Definition 3.3.** Define the agent \( i \)'s consumption and portfolio policy correspondence \( \tilde{x}^i : \Theta^i \times S \times \hat{Q}^o \times \hat{\Theta} \rightarrow C^i \times \Theta^i \) as

\[
\tilde{x}^i(\theta^i, s, \hat{q}, \hat{\theta}) = \arg\max \left\{ u(c^i) + \beta \int_z v^i(\theta^i(s), \hat{q}, \hat{\theta}) \lambda^i(z, dz') \right\}
\]

over all \( (c^i, \theta^i) \in B^i(\theta^i, z, \hat{q}(s)) \);

**Remark 3.4.** Notice that the policy correspondence satisfy

\[
\tilde{x}^i(\theta^i, s, \hat{q}, \hat{\theta}) \subset B^i(\theta^i, z, \hat{q}(s)) \text{ for all } (\theta^i, s, \hat{q}, \hat{\theta}) \in \Theta^i \times S \times \hat{Q}^o \times \hat{\Theta}.
\]

Observe also that when \( \tilde{x}^i \) is a function, then it can be written as \( \tilde{x}^i = (\tilde{c}^i, \tilde{\theta}^i) \) for continuous functions \( \tilde{c}^i : \Theta^i \times S \times \hat{Q}^o \times \hat{\Theta} \rightarrow \mathbb{R} \) and \( \tilde{\theta}^i : \Theta^i \times S \times \hat{Q}^o \times \hat{\Theta} \rightarrow \mathbb{R} \).

\[\text{Recall that } s = (\theta, z).\]
The lemma below and the strict concavity of \( u \) assure that we can assume, without loss of generality, that \( \tilde{x}_i \) is actually a function. To show this, it suffices to note that the optimal portfolio choices are unique within the portfolio reallocations that result in the same pay off.

**Lemma 3.5.** Suppose that \( C_i \times \Theta_i = \mathbb{R}^{J+H} \). Then the correspondence \( \tilde{x}_i \) has a continuous selector.

*Proof:* See Lemma 7.9 in the Appendix.

**Definition 3.6.** We say that the economy has an \( \epsilon \)-approximate recursive equilibrium if there exist continuous functions \( \hat{c}_i : S \to C_i, \hat{\theta}_i : S \to \Theta_i \) for \( i \in I \) and \( \hat{q} : S \to Q \) satisfying for each \( s = (\tilde{\theta}, z) \in S \)

1. \( (\hat{c}_i(s), \hat{\theta}_i(s)) \in \tilde{x}_i(\hat{\theta}_i, s, \hat{q}, \hat{\theta}) \) for all \( i \in I \) and \( s \in S \);

2. \[ ||\sum_{i \in I} \hat{c}_i(s) - e^i(z) - \hat{d}(z) \cdot 1|| \leq \epsilon; \]

3. \[ ||\sum_{i \in I} \hat{\theta}_i(s) - 1|| \leq \epsilon. \]

The next definition provides more details of how the recursive functions can implement an \( \epsilon \)-approximate sequential equilibrium.

**Definition 3.7.** We say that the functions \( \hat{c}_i : S \to C_i, \hat{\theta}_i : S \to \Theta_i \) and \( \hat{q} : S \to Q \) generate the process \((c_r, \theta_r, q_r)_{r \geq 0}\) starting from \( \bar{\theta} \in \Theta \) and \( z \in Z \) if for all \( z^r \in Z^r \)

\[
q_0 = \hat{q}(\tilde{\theta}, z), \quad \theta_0^i = \hat{\theta}_i(\tilde{\theta}, z), \quad c_0^i = \hat{c}_i(\tilde{\theta}, z)
\]

and recursively for \( r \in \mathbb{N} \)

\[
c_r^i(z^r) = \hat{c}_i(\theta_{r-1}(z^{r-1}), z_r) \quad \theta_r^i(z^r) = \hat{\theta}_i(\theta_{r-1}(z^{r-1}), z_r) \quad (5)
\]

for \( i \in I \) and

\[
q_r(z^r) = \hat{q}(\theta_{r-1}(z^{r-1}), z_r). \quad (6)
\]

The next result assures that the recursive equilibrium can actually be used to construct the sequential equilibrium.

**Theorem 3.8.** If \((\hat{c}, \hat{\theta}, \hat{q})\) is an \( \epsilon \)-approximate recursive equilibrium then its implemented process \((c, \theta, q)\) starting from \((\tilde{\theta}, z) \in S\) is an \( \epsilon \)-approximate sequential equilibrium of the economy with initial asset holdings \( \tilde{\theta} \in \Theta \) and initial state of the nature \( z \).

*Proof:* See Theorem 7.8 in the appendix.
4 Existence and Convergence Results

4.1 Existence of an approximate recursive equilibrium

In this section we will demonstrate the existence of approximate recursive equilibrium with state space $S = \Theta \times Z$. For simplicity, we will make a slight change in notation.

**Notation 4.2.** Write $L = \{1, 2, ..., L\}$ for $L = H + J \geq 2$ and for each $\gamma > 0$ the set $Q_\gamma = Q \cap [\gamma, 1]^L \subset \mathbb{R}^L_+$. Consider $X^i = C^i \times \Theta^i \subset \mathbb{R}^L_+$ convex with $0 \in X$ and $Q' = (0, 1]^L$ the set of generalized prices. Define $\hat{X}^i$ as the set of all bounded continuous functions $\hat{x}^i : S \to X^i$ and $\hat{Q}'$ the set of all bounded continuous functions $\hat{q} : S \to Q'$ both endowed with the topology induced by the sup metric. Recall that $\hat{Q}$ is the set of all bounded continuous normalized prices $\hat{q} : S \to Q$. Consider $V^i : X^i \times S \times \hat{X} \times \hat{Q}' \to \mathbb{R}$ a bounded continuous function such that $V^i(\cdot, s, \hat{x}, \hat{q})$ is concave and increasing for all $(s, \hat{x}, \hat{q}) \in S \times \hat{X} \times \hat{Q}'$ and all $i \in I$. Moreover we suppose that $V^i(x^i, s, \hat{x}, \cdot)$ is homogeneous of degree zero for all $(x^i, s, \hat{x}) \in X^i \times S \times \hat{X}$. Write $\hat{w}^i : S \to W^i \subset \mathbb{R}^{J+H}$ as the agent $i$’s contingent endowments with $W^i$ compact convex. Recall that we consider $\|\cdot\|$ as the max norm in $\mathbb{R}^L$, that is, $\|x\| = \max\{|x_1|, ..., |x_n|\}$.

The next definitions will be useful in the main result of this section.

**Definition 4.3.** Define $\hat{v}^i : S \times \hat{X} \times \hat{Q}' \to \mathbb{R}$ by

$$\hat{v}^i(s, \hat{x}, \hat{q}) = \sup \{ V^i(x^i, s, \hat{x}, \hat{q}) : x^i \in X^i \text{ and } \hat{q}(s)x^i \leq \hat{q}(s)\hat{w}^i(s) \} \quad (7)$$

and $\hat{x}^i : S \times \hat{X} \times \hat{Q}' \to \mathbb{R}$ by

$$\hat{x}^i(s, \hat{x}, \hat{q}) = \argmax \{ V^i(x^i, s, \hat{x}, \hat{q}) : x^i \in X^i \text{ and } \hat{q}(s)x^i \leq \hat{q}(s)\hat{w}^i(s) \} \quad (8)$$

**Remark 4.4.** Note that $\hat{v}^i(\cdot, \hat{x}, \cdot)$ and $\hat{x}^i(\cdot, \hat{x}, \cdot)$ are homogeneous of degree zero for all $(s, \hat{x}) \in S \times \hat{X}$.

We will define below the Lipschitz property. This property is related to the level of oscillation of a function. For differentiable functions this means that the function must have bounded derivative.

\[11\text{In the equilibrium, actually, the prices will be normalized, that is belongs to } Q^\circ \subset Q'.\]
**Definition 4.5.** A function \( f : Y \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \) is \( M \)-Lipschitz for \( M \in \mathbb{R}^+ \) if \( ||f(y) - f(y')|| \leq M ||y - y'|| \) for all \( y, y' \in Y \). When \( M \in \mathbb{R}^m^+ \) then we say that \( f = (f_1, ..., f_m) \) is \( M \)-Lipschitz if \( f_k : Y \subset \mathbb{R}^n \rightarrow \mathbb{R} \) is \( M_k \)-Lipschitz for \( k = 1, ..., m \). We write \( L_p(M) \) the space of all \( M \)-Lipschitz functions.

**Remark 4.6.** Notice that a function \( f : Y \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \in L_p(M) \) for \( M \in \mathbb{R}^m^+ \) then \( f \in L_p(||M||) \).

The hypothesis below precludes the possibility of a situation similar to that described by Hellwig (1982) for the case of approximate recursive equilibrium. It states that small variations in asset endowments can not produce arbitrary large fluctuations in asset demand considering that prices fluctuate in a controlled magnitude. Hellwig (1982) points to the impossibility of existence of a stationary function that relates the equilibrium variables, with rational expectations, between two arbitrary consecutive periods providing and example with demand having arbitrary large fluctuation. We give an example below which clarifies this assumption.

**Assumption 4.7.** Suppose that \( X^i = \mathbb{R}^L \). Assume that the correspondence \( \hat{x}^i \) has a continuous selector \( \hat{\delta}^i : S \times \hat{X} \times \hat{Q}' \rightarrow X^i \). In addition the function \( \hat{\delta}^i : \hat{X} \times \hat{Q}' \rightarrow \hat{X}^i \) defined by

\[
\hat{\delta}^i(\hat{x}, \hat{q})(s) = \hat{\delta}^i(s, \hat{x}, \hat{q}) \quad \text{for all } s = (\hat{\theta}, z) \in S \text{ and all } i \in I
\]  

satisfies the following property: there exists \( M_q, M_x > 0 \) such that if \( \hat{q} \in L_p(M_q) \) and \( \hat{x} \in L_p(M_x) \) then \( \hat{\delta}^i(\hat{x}, \hat{q}) \in L_p(M_x) \).

**Example 4.8.** Consider a model with one good and one asset and agents with instantaneous utility function defined by \( u^i(\cdot) \equiv \ln(\cdot) \) for \( i = 1, 2 \). Suppose that there is not exogenous uncertainty, that is, \( Z = \{1\} \) and \( S = \Theta \). Dividends are given by \( \hat{d} \) and there is not good endowments. We must impose that \( C^i \subset \mathbb{R}^o^+ \) and \( \Theta^i \subset \mathbb{R}^o^+ \) because \( u^i \) is defined only for \( \mathbb{R}^o^+ \). The value function satisfies

\[
v^i(\theta^i, \hat{\theta}, \hat{q}, \hat{\theta}) = \sup \left\{ u^i(c^i) + \beta v^i(\theta^i, \hat{\theta}(\hat{\theta}), \hat{q}, \hat{\theta}) : (c^i, \theta^i) \in B^i(\theta^i, \hat{q}(s)) \right\} \quad (10)
\]

and is strictly concave\(^{12}\) on \( \theta^i \). Since \( u^i \) is strictly concave on \( c^i \) then, in this example, the policy correspondence \( \bar{x}^i \) given in definition 3.3 is actually

\(^{12}\)See Stokey et al. (1989) Chapter 4 for more details.
function. Let \( \tilde{\theta}^i : \Theta^i \times \Theta \times \tilde{Q}' \times \tilde{\Theta} \to \Theta^i \) and \( \tilde{c}^i : \Theta^i \times \Theta \times \tilde{Q}' \times \tilde{\Theta} \to \mathcal{C}^i \) be the policy functions as in Remark 3.4. Lemma 7.7 in appendix assures that \((\tilde{c}^i, \tilde{\theta}^i)\) is nonempty and satisfies \((\tilde{c}^i, \tilde{\theta}^i) > 0\). Observe that:

\[
\tilde{c}^i(\tilde{\theta}^i, \tilde{\theta}, \tilde{q}, \tilde{\theta}) = -\tilde{\rho}(\tilde{\theta})\tilde{\theta}^i(\tilde{\theta}, \tilde{q}, \tilde{\theta}) + (\tilde{\rho}(\tilde{\theta}) + \tilde{d})\tilde{\theta}^i
\]

where we write \( \tilde{\theta} \equiv \tilde{q}^a / \tilde{q}^c \). Fix \((\tilde{\theta}^i, \tilde{q}, \tilde{\theta}) \in \Theta^i \times \Theta \times \tilde{Q}' \times \tilde{\Theta} \) with \( \tilde{\theta}^i > 0\).

We claim that

\[
\phi^i(\tilde{\theta}^i, \tilde{q}, \tilde{\theta}) = \frac{\beta(\tilde{\rho}(\tilde{\theta}) + \tilde{d})\tilde{\theta}^i}{\tilde{\rho}(\tilde{\theta})} \quad \text{and} \quad \tilde{c}^i(\tilde{\theta}^i, \tilde{\theta}, \tilde{q}, \tilde{\theta}) = (1 - \beta)\tilde{\rho}(\tilde{\theta}) + \tilde{d})\tilde{\theta}^i.
\]

Indeed, using that \( u'(c) = 1/c \) and applying the Benveniste and Scheinkman Theorem\(^{13}\) in Benveniste and Scheinkman (1979) we conclude that

\[
\partial_1 v^i(\tilde{\theta}^i, \tilde{q}, \tilde{\theta}) = \partial u^i(\tilde{c}^i(\tilde{\theta}^i, \tilde{\theta}, \tilde{q}, \tilde{\theta}))(\tilde{\rho}(\tilde{\theta}) + \tilde{d})
\]

\[
= (\tilde{\rho}(\tilde{\theta}) + \tilde{d})/(1 - \beta)(\tilde{\rho}(\tilde{\theta}) + \tilde{d})\tilde{\theta}^i
\]

\[
= 1/((1 - \beta)\tilde{\theta}^i)
\]

and hence

\[
\partial_2 v^i(\tilde{\theta}^i, \tilde{\theta}, \tilde{q}, \tilde{\theta}) = \partial(\phi^i(\tilde{\theta}^i, \tilde{\theta}, \tilde{q}, \tilde{\theta}), \tilde{q}, \tilde{\theta})
\]

\[
= \tilde{\rho}(\tilde{\theta})/((1 - \beta)\tilde{\rho}(\tilde{\theta}) + \tilde{d})\tilde{\theta}^i
\]

for all \((\tilde{\theta}^i, \tilde{\theta}, \tilde{q}, \tilde{\theta}) \in \Theta^i \times \Theta \times \tilde{Q}' \times \tilde{\Theta} \). Therefore

\[
\tilde{\rho}(\tilde{\theta})\partial_1 v^i(\tilde{c}^i(\tilde{\theta}^i, \tilde{\theta}, \tilde{q}, \tilde{\theta})) = \tilde{\rho}(\tilde{\theta})/((1 - \beta)\tilde{\rho}(\tilde{\theta}) + \tilde{d})\tilde{\theta}^i
\]

\[
= \beta\tilde{\rho}(\tilde{\theta})/((1 - \beta)\beta(\tilde{\rho}(\tilde{\theta}) + \tilde{d})\tilde{\theta}^i)
\]

\[
= \beta\partial_1 v^i(\tilde{\theta}^i, \tilde{\theta}, \tilde{q}, \tilde{\theta}, \tilde{\theta}(\tilde{\theta}), \tilde{\theta}(\tilde{\theta}))
\]

which implies that \( \tilde{\theta}^i \) satisfies the first order condition on \( \theta^i \) of the Bellman Equation (10) replacing \( c^i \) by \( -\tilde{\rho}(\tilde{\theta})\theta^i + (\tilde{\rho}(\tilde{\theta}) + \tilde{d})\theta^i \). The strict concavity of \( u^i \) and of \( v^i \) on the first coordinate is sufficient to conclude our assertion.

Consider \( P \subset \mathbb{R}_{++} \) closed.\(^{14}\) Clearly, the function \( g^i : \Theta \times P \to \mathbb{R}_+ \) defined by \( g^i(\tilde{\theta}^i, \rho) = (1 + d/p)\theta^i \) satisfies \( g^i \in L^p(M') \) for some constant

\(^{13}\)See also Stokey et al. (1989) for more details about this theorem.

\(^{14}\)Note that \( P \) is bounded away from zero.
M', since it has bounded derivative. Therefore, it is possible to find \(M_x = (M^c, M^a)\) and \(M_q\) such that the function

\[
\tilde{\theta} \mapsto \tilde{\theta}^i(\tilde{\theta}, \tilde{\theta}, \tilde{q}, \tilde{\theta}) = \beta g^i(\tilde{\theta}, \tilde{q}^a(\tilde{\theta})/\tilde{q}^c(\tilde{\theta})) \in L^p(M^a)
\]

and that \(\tilde{c}^i \in L^p(M^c)\) by using (11) if \(\tilde{q} \in L^p(M_q)\) and \(\tilde{\theta} \in L^p(M^a)\).

Finally, the following lemmas are useful in the proof of the main result of this section. Its result will be used to construct an operator whose fixed point is the recursive equilibrium.

**Lemma 4.9.** Consider \(Y \subset \mathbb{R}^n\) and \(M, N \in \mathbb{R}^{++}\). Suppose that \(f : Y \to Y\) is such that \(f \in L^p(M)\) and \(g : Y \to Y \in L^p(N)\). Then \(f \circ g \in L^p(MN)\), \(f + g \in L^p(M + N)\).

*Proof:* See Lemma 7.4 in the Appendix.

**Lemma 4.10.** Consider \(E \subset \mathbb{R}^L\), \(\gamma < 1/L\) and let \(\pi : E \to \mathbb{R}\) be defined by \(\pi(\xi) = \max \{q\xi : q \in Q^\gamma\}\). Then given \(\epsilon > 0\) the correspondence

\[
\xi \to \{q \in Q^\gamma : \pi(\xi) - q\xi \leq \epsilon\}
\]

has a selector \(\Delta : E \to Q^\gamma\) with \(\Delta \in L^p(4L/\epsilon)\).

*Proof:* See Lemma 7.6 in the appendix.

The next assumption holds for the model on which \(V^i\) is defined using the value function \(v^i\) as clarified by Theorem 4.13.

**Assumption 4.11.** For each compact \(\tilde{X} \subset \text{Int} X\) there exists \(\gamma > 0\) such that if \(\tilde{q} \in Q\) satisfies \(\tilde{\delta}^i(\tilde{x}, \tilde{q})(S) \subset \tilde{X}^i\) for all \(i \in I\) and some \(\tilde{x} \in \tilde{X}\), then \(\tilde{q}_l(s) \geq \gamma\) for all \(l \in L\) and all \(s \in S\).

**Theorem 4.12.** Suppose Assumptions 4.7 and 4.11 and consider that \(\tilde{\delta}^i\) defined in Assumption 4.7 for \(i \in I\). Then for each fixed \(\epsilon > 0\) there exist continuous functions \(\tilde{x}^i : S \to \mathbb{R}^L_+\) for all \(i \in I\) and \(\tilde{q} : S \to Q^\gamma\) such that \(\tilde{x}^i \in \tilde{\delta}^i(\tilde{x}, \tilde{q})\) for all \(i \in I\) and \(|\sum_{i \in I} (\tilde{x}^i(s) - \tilde{w}^i(s))| \leq \epsilon\) where \(|\cdot|\) is the max norm and \(\tilde{x} = (\tilde{x}^i)_{i \in I}\).

\(^{15}\)Note that \((\tilde{c}^i, \tilde{\theta}^i)\) is independent of \((\tilde{c}^i, \tilde{\theta}^i)\).
Proof: We can suppose without loss of generality that $\varepsilon \leq 1$. Define $X^i$ a compact convex cube containing the compact set $\{x \in \mathbb{R}_+^L : \sum_{i \in \mathcal{I}} x^i \leq \sum_{i \in \mathcal{I}} \max\{\bar{w}^i(s) : s \in S\} + 1\}$ (12) in its interior relative to $\mathbb{R}_+^L$. Write $X = \prod_{i \in \mathcal{I}} X^i$ and let $\delta^i$ be the correspondence given in (9). We can assume that the range of $\delta^i$ is contained in $X^i$. Indeed, if not, choose $m^i = \sup X^i \in \mathbb{R}_+^L$ and define $\delta^i(\hat{x}, \hat{q})(s) = \min\{m^i, \delta^i(\hat{x}, \hat{q})(s)\} \in X^i$ for all $s \in S$. Clearly, $x^i(s) = \delta^i(\hat{x}, \hat{q})$ implies $\hat{q}(s)\hat{x}(s) \leq \hat{q}(s)\hat{w}^i(s)$ and $\delta^i(\hat{x}, \hat{q}) \in Lp(M_x)$ if $\hat{q} \in Lp(M_q)$ and $\hat{x} \in Lp(M_x)$. Thus $\delta^i$ keep all necessary properties to the remainder of the proof since, in the approximate equilibrium $(\hat{x}, \hat{q})$, we must have $\delta^i(\hat{x}, \hat{q}) = \delta^i(\hat{x}, \hat{q}) \in \hat{X}^i$ for all $i \in \mathcal{I}$ by (12). Let $\xi : X \times W \rightarrow \mathbb{R}^L$ be the excess of demand function defined by $\xi(x, w) = \sum_{i \in \mathcal{I}} (x^i - w^i)$. Since $\xi$ is continuous and $X \times W$ is compact, there exists $M' > 0$ such that $||\xi(x, w)|| \leq M'$ for all $(x, w) \in X \times W$. Consider $\gamma'$ as in Lemma 4.11 with respect to the compact set given in (12) and write $N = \max\{16IL(M_x + M_w)/(\gamma'\varepsilon), 1\}$. Choose $\gamma < \min\{\varepsilon\gamma'/2M', 1/4, \gamma'\}$ and $Q'_\gamma = Q' \cap [\gamma, 1]^L$. Define $\hat{Q}'_\gamma = \{\hat{q} \in \hat{Q}' \cap Lp(N) : \hat{q}(S) \subset Q'_\gamma\}$ and $\hat{X}' = \hat{X} \cap Lp(M_x)$. 

Ascoli’s Theorem in Royden (1963) and the compactness of $Q'_\gamma$ and $X$ assure that $\hat{Q}'_\gamma$ and $\hat{X}'$ are compact. Define the function $\Delta : \hat{X}' \rightarrow \hat{Q}'_\gamma$ as $\Delta(\hat{x})(s) = \Delta(\hat{x}(s), \hat{w}(s)))$ for all $s \in S$ where $\Delta : Z \rightarrow Q'_\gamma$ is given in Lemma 4.10 for the error $\gamma'/4$ and the lower bound $\gamma$ on prices. Lemmas 4.9 and 4.10 also guarantees that $\hat{\Delta}(\hat{x}) \in \hat{Q}'_\gamma$ since $\hat{w} \in Lp(M_w)$ implies that $\sum_{i \in \mathcal{I}} \hat{x}^i - \hat{w}^i \in Lp(I(M_x + M_w)))$. 

Write $N' = \min\{M_q/N, 1\}$. Let $T : \hat{X}' \times \hat{Q}'_\gamma \rightarrow \hat{X}' \times \hat{Q}'_\gamma$ be the continuous convex valued correspondence (function) defined by: $T(\hat{x}, \hat{q}) = \prod_{i \in \mathcal{I}} \delta^i(\hat{x}, N'_q \hat{q}) \times \Delta(\hat{x})$.

---

16 In the case of $\varepsilon > 1$ remake all arguments replacing $\varepsilon$ by 1.

17 A cube in $\mathbb{R}^L$ is a product of a compact interval contained in $\mathbb{R}$. Moreover, for a set $X^i \subset \mathbb{R}_+^L$, write $\sup X^i = (\sup X^i)_i \in \mathbb{R}_+^L$ where $X^i \subset \mathbb{R}_+$ is the projection of $X^i$ into the $i$-th coordinate. Observe also that $0 \in X$. 

16

17
Clearly, Assumption 4.7 assures that \( T \) is well defined because \( \hat{q} \in \hat{Q}' \) implies \( N'\hat{q} \in Lp(M_q) \) if \( M_q \leq N \) or \( N'\hat{q} \in Lp(N) \subset Lp(M_q) \) if \( M_q \geq N \) and hence \( \hat{d}^i(x, N'\hat{q}) \in \hat{X}' \) in both cases. Since \( \hat{X}' \times \hat{Q}' \) is a nonempty compact convex space endowed with a locally convex Hausdorff topology and \( T \) has closed graph, we can apply the Kakutani-Fan-Glicksberg Fixed Point Theorem in Aliprantis and Border (1999) Theorem 17.55 to conclude that \( T \) has a fixed point, say, \((\hat{x}, \hat{q})\).

To show the market clearing conditions notice that since \( N'\hat{q}(s)\hat{x}^i(s) \leq N'\hat{q}(s)\hat{w}^i(s) \) then \( N'\hat{q}(s)(\hat{x}^i(s) - \hat{w}^i(s)) \leq 0 \) for all \( s \in S \) and if we add over \( i \in I \) these budget restrictions we get

\[
\hat{q}(s)\hat{\xi}(\hat{x}(s), \hat{w}(s)) \leq 0 \quad \text{for all } s \in S.
\]  
(13)

Suppose that \( \hat{\xi}_l(\hat{x}(s'), \hat{w}(s')) \geq \epsilon \gamma' \) for some \( s' \in S \). Then \( \hat{q} \in \hat{\Delta}(\hat{x}) \) implies that \( \pi_\gamma(\hat{\xi}(\hat{x}(s'), \hat{w}(s'))) - \hat{q}(s')\hat{\xi}(\hat{x}(s'), \hat{w}(s')) \leq \epsilon \gamma'/4 \). Choose \( q' \in Q' \) such that \( q'_l = 1 - (L - 1) \gamma \) and \( q'_k = \gamma \) for all \( k \neq l \). Then

\[
q'\hat{\xi}(\hat{x}(s'), \hat{w}(s')) = q'_l\hat{\xi}_l(\hat{x}(s'), \hat{w}(s')) + \sum_{k \neq l} q'_k\hat{\xi}_k(\hat{x}(s'), \hat{w}(s'))
\]

\[
> (1 - (L - 1) \gamma)\epsilon \gamma' - \sum_{k \neq l} q'_k M'
\]

\[
= (1 - (L - 1) \gamma)\epsilon \gamma' - (L - 1) \gamma M'
\]

\[
> (1/2 - (L - 1) \gamma)\epsilon \gamma'
\]

\[
> \epsilon \gamma'/4.
\]

Where in third and fourth inequalities we use that \( (L - 1) \gamma < \min\{\epsilon \gamma'/(2M'), 1/4\} \). Therefore

\[
0 < q'\hat{\xi}(\hat{x}(s'), \hat{w}(s')) - \epsilon \gamma'/4
\]

\[
\leq \pi_\gamma(\hat{\xi}(\hat{x}(s'), \hat{w}(s'))) - \epsilon \gamma'/4
\]

\[
< \hat{q}(s')\hat{\xi}(\hat{x}(s'), \hat{w}(s'))
\]

\[
\leq 0
\]

which is a contradiction. We have thus proved that

\[
\hat{\xi}_l(\hat{x}(s), \hat{w}(s)) \leq \epsilon \gamma' < \epsilon
\]  
(14)

\footnote{Notice that \( \gamma < 1/L < 1/(L - 1) \).}
for each \( s \in S \) and \( l \leq L \). To show that \( \hat{\xi}(\hat{x}(s), \hat{w}(s)) \geq -\epsilon \cdot 1 \), notice first that, using (14) we can conclude that \( \xi(\hat{x}(s), \hat{w}(s)) \leq \epsilon \cdot 1 \leq 1 \) for \( s \in S \) and all \( i \in I \). Therefore, \( \hat{x}(s) \in \tilde{X} \subset \text{Int} X \) and hence \( \hat{\theta}(s) \geq \gamma^i \) for all \( s \in S \) and all \( l \leq L \) by Assumption 4.11 and by the fact that the correspondence \( \Delta \) have range in the set of normalized prices and hence \( \tilde{\Delta}(\hat{x}) \in \hat{Q} \). Moreover, \( \hat{\theta}(s)\xi(\hat{x}(s), \hat{w}(s)) = 0 \). Indeed, all budget inequalities must bind because \( \xi(\hat{x}(s), \hat{w}(s)) \leq 1 \) implies \( \hat{x}(s) \in \text{Int} X^i \) for all \( i \in I \). Thus, for each \( s \in S \), using equation (14) we get

\[
-\hat{q}(s)\xi(\hat{x}(s), \hat{w}(s)) = \sum_{l \neq k} \hat{q}(s)\xi_l(\hat{x}(s), \hat{w}(s)) \leq \epsilon \gamma^i.
\]

Therefore, using that \( \hat{q}(s) \geq \gamma^i \), we conclude that

\[
\xi(\hat{x}(s), \hat{w}(s)) \geq -\epsilon / \hat{q}(s) \geq -\epsilon
\]

for each \( l \neq L \). To conclude the theorem, notice that \( \hat{\delta}(\hat{x}, N^i \hat{q}) = \hat{\delta}(\hat{x}, \hat{q}) \) by Remark 4.4.

**Theorem 4.13.** Define \( S = \{ (\hat{\theta}, z) \in R^{|I|}_+ \times Z : \sum_{i \in I} \hat{\theta}^i = 1 \} \) and \( \mathcal{J} \) its Borel \( \sigma \)-algebra. Write \( X^i = C^i \times \Theta^i \) with \( \Theta^i = R_+^H \) and \( C^i = R_+^J \). Suppose that \( V^i : X^i \times S \times \hat{Q}^i \times \tilde{X} \to \mathbb{R} \) is given by

\[
V^i(c^i, \theta^i, s, \hat{\theta}, \hat{\theta}) = u(c^i) + \beta \int_Z v^i(\theta^i, (\hat{\theta}(s), z'), \hat{\theta}, \hat{\theta}) \lambda(z, dz')
\]

and satisfies Assumption 4.7, where \( v^i \) is the value function given by (2). Then Theorem 4.12 holds for \( \{ V^i \}_{i \in I} \) where \( \hat{x} := (c, \theta) \) and \( \hat{w}(s) := (\hat{d}(z)\hat{\theta}^i, \hat{\theta}^i) \).

**Proof:** In this case, all hypothesis satisfied by \( V^i \) in Theorem 4.12 are assured. Indeed, Assumption 4.11 is a direct application of Lemma 2.11 and Theorem 3.8 because \( (\hat{x}, \hat{q}) \) implement the optimal choices \( (c^i, \theta^i) \) given \( s = (\hat{\theta}, z) \) such that \( (c^i, \theta^i) \in \hat{\delta}(\theta^i, z, \hat{q}) \) for all \( i \in \mathcal{I} \). The argument used to show optimality of the equilibrium even considering \( C^i \times \Theta^i = R^{J+H} \) is standard.\(^19\) For this, it is enough to use that \( u^i \) is concave and that the optimum belongs to the interior of \( C^i \times \Theta^i \).

\(^{19}\)See Debreu (1959) or Mas-Colell et al. (1995) for further details.
4.14 Convergence to an exact equilibrium

In this section, we describe how the intertemporal variables of the economy could be close to a sequential competitive equilibrium through a possible anticipation of a recursive relation among them. First, we will define the concept of Jordan temporary equilibrium\(^{20}\) similarly to that given in Jordan (1977) where agents can remake their plans in a given period if they fail to coordinate on the same price stream \(\{q_t\}_{t \in \mathbb{N}}\) and show that it can be arbitrarily close to an exact sequential equilibrium with intertemporal consistency. The later would be the competitive equilibrium in which the economy would actually reach up if agents reduce their computational errors gradually. In this case, they have no incentive to deviate from their expectations on future prices and the plans do not change anymore.

Notation 4.15. The index \(t_r\) indicates the variable planned at date \(t\) for \(r \geq 1\) periods ahead. In case \(r = 0\), the index \(t_0\) indicates the current variable in period \(t\).

Definition 4.16. Let \(\bar{\theta}\) be an initial portfolio distribution and for each period \(t\), consider the following measurable\(^{21}\):

1. \(q_i(z_t) := \{q_{tr}(z_t) : Z^r \to Q^s\}_{r \geq 0}\) contingent prices;
2. \(c_i(z_t) := \{c_{tr}(z_t) : Z^r \to C^s\}_{r \geq 0}\) contingent consumption allocation;
3. \(\theta_i(z_t) := \{\theta_{tr}(z_t) : Z^r \to \Theta^s\}_{r \geq 0}\) contingent portfolio allocation.

Then these allocations and prices constitute a Jordan temporary equilibrium for \(E\) if satisfy for all \(z_t \in Z^t\) and all \(t \in \mathbb{N}\):

1. optimality: \((c_i(z_t), \theta_i(z_t)) \in \delta^i(\theta_{t-i,0}(z^{t-1}), z_t, q_{tr}(z_t))\);
2. asset markets clearing: \(\sum_{i \in I} \theta_{tr}(z_t)(z^r) = 1\) for all \(r \in \mathbb{N}\);
3. good markets clearing: \(\sum_{i \in I} c_{tr}(z_t)(z^r) = \hat{d}(z_r) \cdot 1 + e^i(z_r)\) for all \(r \in \mathbb{N}\).

where we adopt by convention that \(\theta_{t_0}(z^0) = \bar{\theta}\).

\(^{20}\)This concept incorporates the possibility of intertemporal inconsistency.
\(^{21}\)When \(r = 0\) consider \((c_{t0}(z^t), \theta_{t0}(z^t), q_{t0}(z^t))\) constant.
Remark 4.17. Note that if there exists \( t \in \mathbb{N} \) such that the Jordan temporary equilibrium satisfies for each \( r \in \mathbb{N} \)

\[
(c_{tr}, \theta_{tr}, q_{tr})(z^t) = (c_{t+r,0}, \theta_{t+r,0}, q_{t+r,0})(z^{t+r})
\]

then the economy eventually achieves a competitive equilibrium with intertemporal consistency, that is, a Magill-Quinze equilibrium with infinite lived assets without short sales.

The result below is an application of Theorem 4.12 and clarify how a competitive equilibrium can be approximated by implemented \( \epsilon_t \)-approximated recursive equilibrium for \( t \in \mathbb{N} \).

**Theorem 4.18.** Consider a sequence of errors \( \{\epsilon_t\}_{t \in \mathbb{N}} \). Then there exists a sequence of \( \epsilon_t \)-approximate recursive equilibrium \( \{c_t, \theta_t, q_t\}_{t \in \mathbb{N}} \) with \( \epsilon_t \leq \epsilon_t \) and a Jordan temporary equilibrium \( \{c_t, \theta_t, q_t\}_{t \in \mathbb{N}} \) with initial distribution \( \tilde{\theta} \) and such that the distance\(^{22}\) from the equilibrium implemented by \( \{c_t, \theta_t, q_t\}_{t \in \mathbb{N}} \) starting from \( (\theta_0(z^{-1}), z_t) \) and \( \{c_{tr}, \theta_{tr}, q_{tr}\}_{t \in \mathbb{N}} \) starting from \( z_t \) is smaller than \( \epsilon_t \) for all \( t \in \mathbb{N} \).

**Proof:** Fix \( t \) and suppose that \( s_t = (\theta_{t-1,0}(z^{-1}), z_t) \) is given. Choose \( \epsilon''_k = 1/k \) for \( k \in \mathbb{N} \). Consider \( \{\hat{c}_k, \hat{\theta}_k, \hat{q}_k\} \) the \( \epsilon_k \)-approximate recursive equilibrium and \( \{c_{tk}, \theta_{tk}, q_{tk}\} \) its implemented \( \epsilon_k \)-approximate sequential equilibrium starting from \( s_t \). Since \( Z \) is finite,\(^{23}\) we can choose, passing to a subsequence, \( \{c_{tk_n}, \theta_{tk_n}, q_{tk_n}\}_{n \in \mathbb{N}} \) such that

\[
\{c_t, \theta_t, q_t\} := \lim_{n \to \infty} \{c_{tk_n}, \theta_{tk_n}, q_{tk_n}\}.
\]

We claim that the limit prices \( \{q_t\}_{t \in \mathbb{N}} \) are positive. Indeed, fix \( z^t \in Z^t \). Since \( q_0(z^t) \in Q \), there exists \( l \in \mathcal{L} \) such that \( q_0(z^t) \neq 0 \). Moreover, for \( \epsilon_{kn} \leq 1/2 \), we must have \( \sum_i \theta_{l-1,0, kn}^{i} (z^{-1}) \geq 1/2 \) for all \( n \in \mathbb{N} \) by the approximate market clearing conditions, and hence, \( \sum_i \theta_{l-1,0}^i (z^{-1}) > 0 \). Therefore, there exists \( j \in \mathcal{I} \) such that \( \theta_{l-1,0}^j (z^{-1}) > 0 \). Using Proposition 2.10 we conclude that the value function \( \hat{v}^j \) given in Definition 2.8 is continuous on \( (\theta_{l-1,0}^j (z^{-1}), z_t, q_t) \). Thus, \( \{c_t^j, \theta_t^j\} \in \delta^j(\theta_t^j (z^{-1}), z_t, q_t) \) and

\(^{22}\)See Aliprantis and Border (1999) Section 3.9 about the metric in the Cartesian product.

\(^{23}\)Using the product topology, the Tychonoff Product Theorem and the fact that \( \{c_{tk}, \theta_{tk}, q_{tk}\}_{k \in \mathbb{N}} \) is contained in a compact set.
\((c_r^t, \theta_r^t) \in \text{Int} C^j \times \Theta^j\) for all \(r \in \mathbb{N}\) which implies that \(q_r(z^t) > 0\) for all \(r \in \mathbb{N}\). Therefore the profile \(\{c_t, \theta_t, q_t\}_{t \in \mathbb{N}}\) is actually a Jordan temporary equilibrium. To conclude the proof, for a fixed \(t \in \mathbb{N}\), pick \(n \in \mathbb{N}\) such that \(\epsilon_n := 1/k_n \leq \epsilon'_t\) and that \(d((c_{tk_n}, \theta_{tk_n}, q_{tk_n}), (c_t, \theta_t, q_t)) \leq \epsilon'_t\). Choosing \(\epsilon_t = \epsilon_n\) and \((\hat{c}_t, \hat{\theta}_t, \hat{q}_t) = (\hat{c}_{kn}, \hat{\theta}_{kn}, \hat{q}_{kn})\), we get the desired property using induction on \(t\) with initial conditions \(\{s_t\}_{t \in \mathbb{N}}\).

Remark 4.19. Theorem 4.18 allow us to conclude that if agents update the recursive statistics with arbitrary small errors then, in the long run, they are close to the behavior of competitive equilibrium models presented in Radner (1972) or Magill and Quinzii (1996) with intertemporal consistency, even basing their choices in anticipation of approximate future prices using continuous statistics with minimal state space to compute the equilibrium. We conclude that the transition functions of the recursive equilibrium represent an important instrument of coordination among agents if the measurement errors become small enough to avoid that agents deviate from their previous plans.

5 Numerical Method

The numerical method used here is similar to that found in Judd (1998). Basically, we proceed iterating functions recursively to find the solution of a certain functional equation as a fixed function on the limit\(^{24}\).

To clarify the method, given a price \(\hat{q}\) and a transition \(\hat{\theta}\), write

\[\hat{\theta}'(\bar{\theta}^i, \bar{\theta}, z, \hat{q}, \hat{\theta}) = (\bar{\theta}'(\bar{\theta}^i, \bar{\theta}, z, \hat{q}, \hat{\theta}))_{i \in \mathcal{I}}\] for all \((\bar{\theta}, z) \in S\)

where \(\bar{\theta}^i : \Theta^i \times \Theta \times Z \times \hat{Q}^o \times \hat{\Theta} \rightarrow \Theta^i\) is the argmax of Bellman Equation (2). We compute first the value function \(v^i(\cdot, \cdot, \hat{q}, \hat{\theta})\) on a finite set called \(\text{gridS} \subset S\) using the standard Bellman Method iterating value functions given an initial function \(v^i_0\). After that, we compute the asset excess demand

\[\hat{\xi}^a(\bar{\theta}, z) = \sum_{i \in \mathcal{I}} \bar{\theta}^i(\bar{\theta}^i, \bar{\theta}, z, \hat{q}, \hat{\theta}) - 1\] for all \((\bar{\theta}, z) \in S\),

using the argmax of the value function \(v^i(\cdot, \cdot, \hat{q}, \hat{\theta})\) on \(\text{gridS}\). Choosing a price \(\hat{q}' = \hat{q} + \Delta \hat{q}' \in \hat{Q}^o\) where \(\Delta \hat{q}\) is an increment proportional to

\(^{24}\text{On the sup norm.}\)
Figure 1: Graphics of $\bar{\theta}^1 \mapsto \hat{\theta}^1(\bar{\theta}^1, 1-\bar{\theta}^1)$ and $\bar{\theta}^2 \mapsto \hat{\theta}^2(1-\bar{\theta}^2, \bar{\theta}^2)$ for $e^1(z) = p1$ and $\lambda^2(z) = p2$ for all $s \in \{z_1, z_2\}$.

excess demand function $\hat{\xi}^a$, with the proportionality constant chosen appropriately to ensure the speed of convergence; we compute again $\hat{\theta}^a = \{\hat{\theta}^a(\bar{\theta}, \theta, z, \hat{\theta}', \hat{q}')\}_{i \in I}$ for all $(\bar{\theta}, z) \in gridS$ and repeat the previous step until the desired precision. Notice that the convergence of the algorithm implies that the increment $\Delta\hat{q}'$ is near zero and therefore the excess demand $\hat{\xi}^a$ also tends to zero in the limit.

We suppose that agents have homogeneous beliefs $\lambda^i = \lambda$ and heterogeneous instantaneous income $e^i : Z \to \mathbb{R}^+$ depending on the states of the nature. The budget set becomes for $i = 1, 2$

$$
B^i(\theta^i, z, q) = \{(c^i, \theta^i) \in C^i \times \Theta^i : q^c c^i + q^a \theta^i \leq (q^a + q^c \hat{d}(z))\theta^i + q^c e^i(z)\}
$$

and we choose $e^1(z_1) = 1$, $e^1(z_2) = 1$, $e^2(z_1) = 1$ and $e^2(z_2) = 2$ that is, this example includes aggregate risk on the income. Type 1 has initial asset endowment $\theta^1_0 = 0.1$ and type 2 has initial asset endowment $\theta^1_0 = 0.9$. As a result we get linear affine policy functions and the dynamics of the simulations shows that all agents survive and the consumption and asset stream eventually becomes a stationary process. Figure 1 shows the policy functions. It is clear graphically that these functions are linear affine.

Figure 2 shows that prices are independent of the income distribution and figure 3 shows that average consumption and asset stream becomes stationary over time.
Figure 2: Graphics of $\bar{\theta}^1 \mapsto \hat{q}(\bar{\theta}^1, 1 - \bar{\theta}^1, z_1)$ and $\bar{\theta}^1 \mapsto \hat{q}(\bar{\theta}^1, 1 - \bar{\theta}^1, z_2)$

Figure 3: Graphics of the asset average stream $\sum_{r \leq 5000} (c_{ir}^i)(z_r^i)/5000$ for $i = 1, 2$ on the left, $\sum_{r \leq 5000} (\theta_{2r}^i)(z_r^i)/5000$ for $i = 1, 2$ on the right, $r = 5000$ trajectories of $(z_r^1, z_r^2, ..., z_r^T)$, $T = 100$ and $\lambda^2(z) = p = (0.4, 0.6)$ for all $s \in \{z_1, z_2\}$. 
6 Conclusion

There exists a recursive equilibrium with minimal state space implementing a sequential equilibrium arbitrarily closed to a Magill Quinzii competitive equilibrium. Therefore, if agents update the recursive statistics with arbitrary small errors then, in the long run, they are close to the behavior of competitive equilibrium models presented in Radner (1972) or Magill and Quinzii (1996) with intertemporal consistency, even basing their choices in anticipation of approximate future prices using continuous statistics with minimal state space to compute the equilibrium. We conclude that the transition functions of the recursive equilibrium represent an important instrument of coordination among agents if the measurement errors become small enough to avoid that agents deviate from their previous plans. Moreover, it is possible to compute the approximate recursive equilibrium without using the first order conditions of optimality, through a method of iterating functional equations.

7 Appendix

Notation 7.1. Recall that \( \mathcal{L} = \{1, 2, \ldots, L\} \). Consider \( Z, X^i \subset \mathbb{R}^L_+ \), \( Q = \{q \in \mathbb{R}^L_+ : \sum_{t \in \mathcal{L}} q_t = 1\} \), \( Q^\circ = Q \cap \mathbb{R}^L_{++} \), \( Q_\gamma = Q \cap [\gamma, 1]^L \) and \( S \) a topological space. Define \( \hat{X}^i \) as the set of all bounded continuous functions \( \hat{x}^i : S \rightarrow X^i \) and \( \hat{Q} \) the set of all bounded continuous functions \( \hat{q} : S \rightarrow Q \) both endowed with the topology induced by the sup metric. Define \( \hat{Q}^\circ \) analogously. The absence of the upper index stands for the Cartesian product on \( i \in \mathcal{I} \) and we write \( 1 = (1, 1, \ldots, 1) \in \mathbb{R}^L_+ \).

Lemma 7.2. Suppose that \( X^i \subset \mathbb{R}^L_+ \) is a compact convex set with \( 0 \in X^i \) and that \( W^i = \mathbb{R}^L_+ \). Let \( \tilde{B}^i : W^i \times Q^\circ \rightarrow X^i \) be the budget correspondence defined by

\[
\tilde{B}^i(w^i, q) = \{ x^i \in X^i : qx^i \leq qw^i \}.
\]

Then \( \tilde{B}^i \) is continuous.

Proof: If \( X^i = \{0\} \) is trivial. Suppose that \( X^i \neq \{0\} \). The upper hemi-continuity follows from the fact that \( \tilde{B}^i \) has closed graph and compact range space. To show the lower hemi-continuity, let \( (w^i_n, q_n) \in A^i \) converging to \( (\bar{w}^i, \bar{q}) \in A^i \) as \( n \rightarrow \infty \) and \( \bar{x}^i \in \tilde{B}^i(\bar{w}^i, \bar{q}) \).
Suppose first that \( \tilde{q}w^i > 0 \). Then there exists an open set \( O \) of \( A_i \) containing \( (\tilde{w}^i, \tilde{q}) \) such that \( qw^i > 0 \) for all \( (w^i, q) \in O \). Let \( \text{Int} \tilde{B}^i : O \to X^i \) be the correspondence defined by
\[
\text{Int} \tilde{B}^i(w^i, q) = \{ x^i \in X^i : qx^i < qw^i \}.
\]
Since \( 0 \in X^i \), \( \text{Int} \tilde{B}^i \) is nonempty on the set \( O \) and \( X^i \) is convex, then \( \tilde{B}^i(w^i, q) = \text{cl}[\text{Int} \tilde{B}^i(w^i, q)] \) for all \( (w^i, q) \in O \). Clearly, \( \text{Int} \tilde{B}^i \) has open graph. Therefore, using that an open graph correspondence is lower hemicontinuous and that the closure of a lower hemicontinuous correspondence is lower hemicontinuous, we conclude that \( \tilde{B}^i \) is lower hemicontinuous on \( O \) and hence there exists an \( N \subseteq \mathbb{N} \) and a sequence \( x_n^i \in \tilde{B}^i(w_n^i, q_n) \) for each \( n \in N \) such that \( x_n^i \to \bar{x}^i \) as \( n \to \infty \).

If \( \tilde{q}w^i = 0 \) then \( \bar{x}^i = 0 \). Since \( \tilde{q}_1 > 0 \), \( 0 \in X^i \) and \( X^i \) is convex non degenerated, there exists \( N \subseteq \mathbb{N} \) such that \( q_{1n} > 0 \) and \( (q_n w_n^i/q_{1n}, 0) \in X^i \) for \( n \in N \). Choose the sequence \( x_{1n}^i = q_n w_n^i/q_{1n} \) and \( x_{1n}^i = 0 \) for \( l > 1 \) and \( n \in N \). Then \( x_{1n}^i = q_n w_n^i/q_{1n} \to \tilde{q}w^i/q_1 = 0 \) and hence \( x_n^i \to \bar{x}^i = 0 \) as \( n \to \infty \). Moreover, by construction, \( x_n^i \in \tilde{B}^i(w_n^i, q_n) \) for each \( n \in N \).

Lemma 7.3. Consider \( \widetilde{Y}, Y \) metric spaces with \( \widetilde{Y} \) compact and \( \widetilde{Y} \times Y \) endowed with the product topology. Suppose that \( f : \widetilde{Y} \times Y \to \mathbb{R}^L \) is continuous. Let \( C(\widetilde{Y}, \mathbb{R}^L) \) be the space of all bounded continuous functions \( h : \widetilde{Y} \to \mathbb{R}^L \) endowed with the sup norm. Then the function \( g : Y \to C(\widetilde{Y}, \mathbb{R}^L) \) defined by \( g(y)(\cdot) = f(\cdot, y) \) is continuous.

Proof: Consider a sequence \( y_n \to y \) and fix \( \epsilon > 0 \). Then the set \( Y' = \{y_n\}_{n \in \mathbb{N}} \cup \{y\} \) is compact. Therefore, \( f \) is uniformly continuous on \( \widetilde{Y} \times Y' \) and hence the exists \( \delta > 0 \) such that for all \( \bar{y}, \bar{y}' \in \widetilde{Y} \)
\[
d((\bar{y}', y'), (\bar{y}, y)) < \delta \text{ and } y' \in Y' \text{ implies } \|f(\bar{y}', y') - f(\bar{y}, y)\| \leq \epsilon.
\]
Assuming that \( d((\bar{y}', y'), (\bar{y}, y)) = d_Y(\bar{y}', \bar{y}) + d_Y(y', y) \) then choosing \( n_0 \) such that \( n \geq n_0 \) implies \( d_Y(y_n, y) < \delta \) we conclude that \( n \geq n_0 \) implies \( d((\bar{y}, y_n), (\bar{y}, y)) < \delta \) for all \( y \in \widetilde{Y} \) and hence \( \|f(\bar{y}, y_n) - f(\bar{y}, y)\| \leq \epsilon \) for all \( \bar{y} \in \widetilde{Y} \). Thus \( f(\cdot, y_n) \) converges uniformly to \( f(\cdot, y) \). \( \square \)

\(^{25}\)Recall that we are using the relative topology.

\(^{26}\)To see the inclusion \( \tilde{B}^i(w^i, q) \subset \text{cl}[\text{Int} \tilde{B}^i(w^i, q)] \), given \( x^i \in \tilde{B}^i(w^i, q) \) notice that if we choose \( \bar{x}^i \in \text{Int} \tilde{B}^i(w^i, q) \) then \( x_n^i := (1 - 1/n)x^i + \bar{x}^i/n \in \text{Int} \tilde{B}^i(w^i, q) \) and \( x_n^i \to x^i \). Thus \( x^i \in \text{cl}[\text{Int} \tilde{B}^i(w^i, q)] \).

\(^{27}\)The set \( N \) is chosen such that \( (w_n^i, q_n) \in O \) for each \( n \in N \).

\(^{28}\)Clearly, this metric induces the product topology on \( \widetilde{Y} \times Y \).
Lemma 7.4. Consider $Y \subset \mathbb{R}^n$. Suppose that $f : Y \to Y$ and $g : Y \to Y$ satisfy $f \in \operatorname{Lp}(M_f)$ and $g \in \operatorname{Lp}(M_g)$. Then $f \circ g \in \operatorname{Lp}(M_f M_g)$. Moreover, if $f \leq \nu_f$ and $\gamma_g \leq g \leq \nu_g$ for $n = 1$ and $\nu_f, \nu_g, \gamma_g > 0$ then $h = f/g \in \operatorname{Lp}(\nu_g M_f + \nu_f M_g)/\gamma_g^2$, the sum $f + g \in \operatorname{Lp}(M_f + M_g)$ and the product $fg \in \operatorname{Lp}(\nu_g M_f + \nu_f M_g)$.

Proof: Fix $y, y' \in Y$. Thus $||f(g(y)) - f(g(y'))|| \leq M_f ||g(y) - g(y')|| \leq M_f M_g ||y - y'||$. Moreover, for $n = 1$

$$
|f(y)g(y') - f(y')g(y)| \leq |f(y)g(y') - f(y')g(y')
+ f(y')g(y') - f(y')g(y)|
\leq |g(y')||f(y) - f(y')|
+ |f(y')||g(y) - g(y')|
\leq (\nu_g M_f + \nu_f M_g)|y - y'|.
$$

Therefore,

$$
|h(y) - h(y')| = |f(y)g(y') - f(y')g(y')|/|g(y)g(y')|
\leq |y - y'||(\nu_g M_f + \nu_f M_g)/|g(y)g(y')|
\leq (\nu_g M_f + \nu_f M_g)|y - y'|/\gamma_g^2.
$$

Furthermore,

$$
|f(y)g(y) - f(y')g(y')| \leq |f(y)g(y) - f(y')g(y)
+ f(y')g(y) - f(y')g(y')|
\leq |g(y)||f(y) - f(y')|
+ |f(y')||g(y) - g(y')|
\leq (\nu_g M_f + \nu_f M_g)|y - y'|.
$$

Lemma 7.5. Define $m : \mathbb{R}^L \to \mathbb{R}$ by $m(\xi) = \max\{\xi_k : k \in \mathcal{L}\}$. Then $m \in \operatorname{Lp}(1)$.

Proof: Take any $\xi_k$ such that $\xi_k = m(\xi)$. Then $m(\xi) = \xi_k \leq |z - z'| + \xi_k' \leq |z - z'| + m(\xi')$ and hence $m(\xi) - m(\xi') \leq |z - z'|$. By other hand, choosing $\xi_k'$ such that $\xi_k' = m(\xi')$, then $m(\xi') = \xi_k' \leq |z - z'| + \xi_k \leq m(\xi)$ and thus $|m(\xi) - m(\xi')| \leq |z - z'|$. Therefore, $m \in \operatorname{Lp}(1)$. $\Box$
Lemma 7.6. Consider \( E \subset \mathbb{R}^L \), \( \gamma < 1/L \) and let \( \pi_\gamma : E \to \mathbb{R} \) be defined by 
\( \pi_\gamma(\xi) = \max \{ q\xi : q \in Q_\gamma \} \). Then given \( \epsilon > 0 \) the correspondence

\[
\xi \to \{ q \in Q_\gamma : \pi_\gamma(\xi) - q\xi \leq \epsilon \}
\]

has a selector \( \Delta : E \to Q_\gamma \) with \( \Delta \in \text{Lp}(4L/\epsilon) \).

**Proof:** Define \( \pi : E \to \mathbb{R} \) by \( \pi(\xi) = \max \{ q\xi : q \in Q \} \) and \( f_\epsilon : \mathbb{R}^2 \to \mathbb{R}_+ \) by \( f_\epsilon(y) = \max\{1 + (y_1 - y_2)/\epsilon, 0\} \). Then \( f_\epsilon \in \text{Lp}(2/\epsilon) \) by Lemmas (7.4) and (7.5). Moreover, \( \pi(\xi) = m(\xi) \) for all \( \xi \in E \). Define\(^{29} \) \( g : E \to \mathbb{R}_+ \) by \( g(\xi) = \sum_{l \in \mathcal{L}} f_\epsilon(\xi_l, m(\xi)) \). Then \( g \in \text{Lp}(2L/\epsilon) \) by Lemma (7.4) and using that \( g(\xi) \geq 1 \) for all \( \xi \in E \), since \( f_\epsilon(m(\xi), m(\xi)) = 1 \).

Write \( \bar{q}_l = (0, \ldots, 1, \ldots, 0) \) with 1 in the \( l \)-th coordinate. Define \( \Delta(\xi) : E \to Q \) by

\[
\Delta(\xi) = \frac{1}{g(\xi)} \sum_{l \in \mathcal{L}} f_\epsilon(\xi_l, m(\xi))\bar{q}_l \in Q.
\]

Since \( f_\epsilon(\xi_l, m(\xi)) \leq 1 \) and \( 1 \leq g(\xi) \leq L \) for all \( \xi \in E \), then Lemmas (7.4) and (7.5) assure that \( \Delta \in \text{Lp}(4L/\epsilon) \). To show that \( \pi(\xi) - \Delta(\xi) \xi \leq \epsilon \), write \( \mathcal{L}(\xi, \epsilon) = \{ l \in \mathcal{L} : m(\xi) - \xi_l \leq \epsilon \} \) and \( \Delta_l(\xi) \) the \( l \)-th coordinate of \( \Delta(\xi) \) for each \( \xi \in E \). Thus \( \Delta_l(\xi) = 0 \) if \( l \notin \mathcal{L}(\xi, \epsilon) \) and

\[
\pi(\xi) - \Delta(\xi)z = \sum_{l \in \mathcal{L}} \Delta_l(\xi)(\pi(\xi) - \xi_l)
= \sum_{l \in \mathcal{L}(\xi, \epsilon)} \Delta_l(\xi)(m(\xi) - \xi_l) \leq \epsilon.
\]

To complete the proof, first note that if we define \( \gamma' = L\gamma < 1 \), \( \bar{q} = (1/L, \ldots, 1/L) \in Q \) and \( Q' = \{(1 - \gamma')q + \gamma'\bar{q} : q \in Q\} \) then \( Q_\gamma = Q' \). Indeed, \( Q' \subset Q_\gamma \) is trivial since \( \gamma'/L = \gamma \) and \( Q \) is convex. If \( q' \in Q_\gamma \) then \( q' \in Q \) and \( q = (q' - \gamma'\bar{q})/(1 - \gamma') \in Q \) because \( q_l = (q'_l - \gamma'/L)/(1 - \gamma') = (q'_l - \gamma)/(1 - \gamma') \geq 0 \) and \( \sum_{l \in \mathcal{L}} q_l = \sum_{l \in \mathcal{L}} (q'_l - \gamma'/L)/(1 - \gamma') = 1 \). This implies that \( q' = (1 - \gamma')q + \gamma'\bar{q} \in Q' \). Therefore

\[
\pi_\gamma(\xi) = \max \{ q\xi : q \in Q_\gamma \}
= \max \{ (1 - \gamma')q\xi + \gamma'\bar{q}\xi : q \in Q \}
= (1 - \gamma')\pi(\xi) + \gamma'\bar{q}\xi.
\]

\(^{29}\)Observe that \( f_\epsilon(\xi_l, m(\xi)) \leq 1 \) for all \( \xi \in E \).
Thus defining $\Delta'(\xi) = (1 - \gamma') \Delta(\xi) + \gamma' \hat{q} \in Q_\gamma$ we get

$$\pi'_{\gamma}(\xi) - \Delta'(\xi)z = (1 - \gamma')(\pi(\xi) - \Delta(\xi)z) \leq (1 - \gamma')(\pi(\xi) - \Delta(\xi)z) \leq \epsilon$$

for all $\xi \in E$. Clearly, $\Delta' \in L^p((1 - \gamma')4L/\epsilon) \subset L^p(4L/\epsilon)$.

\[\Box\]

**Lemma 7.7.** Suppose that $C^i$ is bounded from above and $Z = \{1\}$. Let $\tilde{\theta}^i : \Theta^i \times \Theta \times \hat{Q}^o \times \hat{\Theta} \to \Theta^i$ and $\bar{\theta}^i : \Theta^i \times \Theta \times \hat{Q}^o \times \hat{\Theta} \to C^i$ be the policy functions as in Definition 3.3. Then $(\tilde{c}^i, \tilde{\theta}^i)$ is nonempty, $\bar{c}^i(\tilde{\theta}^i, \tilde{\beta}, \hat{q}, \bar{\theta}) > 0$ and $\bar{\theta}^i(\bar{\theta}^i, \hat{q}, \bar{\theta}) > 0$ for all $(\bar{\theta}^i, \hat{q}, \bar{\theta}) \in \Theta^i \times \Theta \times \hat{Q}^o \times \hat{\Theta}$ with $\bar{\theta}^i > 0$.

**Proof:** Recall that in Example 4.8, we must impose that $C^i \subset \mathbb{R}_{++}$ and $\Theta^i \subset \mathbb{R}_{++}$ because $u^i$ is defined only for $\mathbb{R}_{++}$. We can suppose that $u^i$ is bounded from above. Indeed, since consumption price is strictly positive and endowments are bounded, we can assume that the feasible consumptions are bounded from above. Then, under the Bellman operator. Therefore, the value function $v^i$ belongs to it and hence $v^i \leq M_c/(1 - \beta)$. Write $g^i(c^i, \theta^i) = u^i(c^i) + \beta v^i(\theta^i, \hat{q}, \bar{\theta})$. This implies that the supremum of $g^i$ on $B^i(\theta^i, \hat{q}(\bar{\theta}))$ is reached at a positive level of consumption. Indeed, suppose that $B^i(\theta^i, \hat{q}(\bar{\theta})) \cap ((0, 1/n] \times \mathbb{R}_+)$ does not contain an optimum point of $g^i$ on $B^i(\theta^i, \hat{q}(\bar{\theta}))$ for all $n \in \mathbb{N}$ and fix $(\tilde{c}^i, \bar{\theta}^i) \in B^i(\theta^i, \hat{q}(\bar{\theta}))$ with $\tilde{c}^i > 0$. Then there exists $(c^i_n, \bar{\theta}^i_n) \in B^i(\theta^i, \hat{q}(\bar{\theta}))$ with $c^i_n \to 0$ and such that $g^i(c^i_n, \bar{\theta}^i_n) \geq g^i(\tilde{c}^i, \bar{\theta}^i)$. But this is a contradiction since $v^i \leq M_c/(1 - \beta)$ and $\lim_{n \in \mathbb{N}} v^i(c^i_n) = -\infty$ which implies that $\lim_{n \in \mathbb{N}} g^i(c^i_n, \theta^i_n) = -\infty$.

To show that $\tilde{\theta}^i(\theta^i, \cdot, \cdot, \cdot) > 0$ for each $\theta^i > 0$ first observe that $v^i$ is unbounded from below. Indeed, let $N > 0$ and consider $\tilde{c}^i$ such that $u^i(\tilde{c}^i) \leq -N - M_c/(1 - \beta)$. Let $\theta^i$ such that $(\tilde{c}^i, \theta^i) \in B^i(\theta^i, \hat{q}(\bar{\theta}))$ implies $\tilde{c}^i \leq \bar{\theta}^i$. Then

$$v^i(\tilde{\theta}^i, \tilde{\beta}, \hat{q}, \bar{\theta}) = \sup \left\{ u^i(c^i) + \beta v^i(\theta^i, \hat{q}(\bar{\theta}), \bar{\theta}) : (\tilde{c}^i, \theta^i) \in B^i(\tilde{\theta}^i, \hat{q}(s)) \right\}$$

$$\leq -N - M_c/(1 - \beta) + \beta M_c/(1 - \beta)$$

$$\leq -N$$

\[\text{30Recall that } V \text{ the space of all bounded value functions } \bar{v}^i : \Theta^i \times Y \times \hat{Q}^o \times \hat{\Theta} \to \mathbb{R} \text{ endowed with the sup norm}\]
Therefore, the argument remaining is analogous to that previously done for consumption supposing by contradiction that \( B^i(\theta^i, \hat{q}(\bar{\theta})) \cap ([\mathbb{R}_+ \times (0, 1/n]) \) does not contain an optimum point of \( g^i \) on \( B^i(\theta^i, \bar{q}(\bar{\theta})) \) and that \( u^i \leq M_c. \)

**Theorem 7.8.** If \((\bar{c}, \bar{\theta}, \bar{q})\) is an \( \epsilon \)-approximate recursive equilibrium then its implemented process \((\bar{c}, \theta, \bar{q})\) starting from \((\bar{\theta}, z) \in \bar{\Theta} \times Z\) is an \( \epsilon \)-approximate sequential equilibrium of the economy with initial asset holdings \( \bar{\theta} \in \bar{\Theta} \).

**Proof:** Since the market clearing conditions comes directly from the definition of the recursive equilibrium, it is sufficient to prove that \((\hat{c}^i, \hat{\theta}^i) \in \delta^i(\bar{\theta}^i, z, \bar{q})\) for all \( z \in Z\) and all \( i \in \mathcal{I}\). Write \( s = (\bar{\theta}, z)\), let \((c^i, \theta^i) \in F^i(\bar{\theta}^i, z, \bar{q})\) be a feasible plan and define

\[
U^i_t(c^i, z) = u^i(c^i_0) + \sum_{\tau=1}^r \beta^\tau u^i(c^i_\tau(z^\tau))\mu^i(z, dz^\tau).
\]

For simplicity, it is convenient to omit the variables \( \bar{q}, \bar{\theta} \) of the value function. Therefore,

\[
v^i(\bar{\theta}^i, s) \overset{\text{def}}{=} \sup \left\{ u^i(c^i) + \beta \int_Z v^i(\theta^i, \hat{\theta}(s), z_1)\lambda^i(z)dz_1 \right\}
\]

(16)

\[
\geq u^i(c^i_0) + \beta \int_Z v^i(\theta^i_0, \hat{\theta}(s), z_1)\lambda^i(z)dz_1.
\]

where the sup in the first equation is over all \((c^i, \theta^i) \in B^i(\bar{\theta}^i, z, \bar{q}(s))\). The inequality above comes from the fact that \((c^i, \theta^i)\) is feasible\(^{31}\) and hence we have that \((c^i_0, \theta^i_0) \in B^i(\bar{\theta}^i, z, \bar{q}) = B^i(\bar{\theta}^i, z, \bar{q}(\bar{\theta}, z))\) because the price recursive relation (6) in definition 3.7.

Using the Bellman Equation again, we have that for each \( z_1 \in Z\)

\[
v^i(\theta^i_0, \hat{\theta}(s), z_1) = \sup \left\{ u^i(c^i) + \beta \int_Z v^i(\theta^i, \hat{\theta}(s), z_1, z_2)\lambda^i(z_1, dz_2) \right\}
\]

\[
\geq u^i(c^i_1(z_1)) + \beta \int_Z v^i(\theta^i_1(z_1), \hat{\theta}(\hat{\theta}(s), z_1), z_2)\lambda^i(z_1, dz_2).
\]

where the sup in the first equation is over all \((c^i, \theta^i) \in B^i(\theta^i_0, z_1, \bar{q}(\hat{\theta}(s), z_1))\). The inequality above comes from the fact that \((c^i, \theta^i)\) is feasible and hence

\(^{31}\)That is, \((c^i, \theta^i) \in F^i(\theta^i, z, q)\)
Indeed the recursive relations in definition 3.7 implies that 
\( \hat{\theta}(s) = \hat{\theta}(\hat{\theta}, z) = \hat{\theta}_0 \) and 
\( \hat{q}(\hat{\theta}(s), z_1) = \hat{q}(\hat{\theta}_0, z_1) = \hat{q}_1(z_1) \). Replacing the previous inequalities\(^{32}\) of 
\( v^i(\hat{\theta}_0, \hat{\theta}(s), z_1) \) in (16) then
\[
v^i(\bar{\theta}, s) \geq u^i(c_0^i) + \beta \int_Z u^i(c_1^i(z_1)) \lambda^i(z, dz_1)
+ \beta^2 \int_Z \int_Z v^i(\theta^i_1(z_1), \hat{\theta}(\hat{\theta}(s), z_1), z_2) \lambda^i(z_1, dz_2) \lambda^i(z, dz_1)
= u^i(c_0^i) + \beta \int_Z u^i(c_1^i(z_1)) \mu_1^i(z, dz_1)
+ \beta^2 \int_{Z^2} v^i(\theta^i_1(z_1), \hat{\theta}(\hat{\theta}(s), z_1), z_2) \mu_2^i(z, dz^2)
= U^i_1(c^i, z) + \beta^2 \int_{Z^2} v^i(\theta^i_1(z_1), \hat{\theta}_1(z_1), z_2) \mu_2^i(z, dz^2)
\]
where in the previous inequality we use that 
\( \hat{\theta}(\hat{\theta}(s), z_1) = \hat{\theta}(\hat{\theta}_0, z_1) = \hat{\theta}_1(z_1) \).
It follows from induction on \( r \) that
\[
v^i(\bar{\theta}_r, s) \geq U^i_{r-1}(c^i, z) + \beta^r \int_{Z^r} v^i(\theta^i_{r-1}(z_{r-1}), \hat{\theta}_{r-1}(z_{r-1}), z_r) \mu_1^i(z, dz_r).
\]
Taking the limit as \( r \to \infty \) and using that \( v^i \) is bounded we get 
\( v^i(\bar{\theta}, s) \geq U^i(c^i, z) \) for all \( (c^i, \theta^i) \) \in \( F^i(\bar{\theta}, z, \hat{q}) \) since \( (c^i, \theta^i) \) was chosen arbitrarily. Therefore we conclude by (1) that 
\( v^i(\bar{\theta}, s, \hat{q}, \hat{\theta}) \geq v^i(\bar{\theta}, z, \hat{q}) \).

Let \( \bar{\theta} : \Theta^i \times S \times \hat{Q} \times \hat{\Theta} \to \Theta^i \) and \( \bar{c}^i : \Theta^i \times S \times \hat{Q} \times \hat{\Theta} \to C^i \) be the policy functions according to Definition 3.3. Definitions 3.6 and 3.7 imply that 
\( (\bar{c}_0^i, \bar{\theta}_0^i) = (c^i(\bar{\theta}, s, \hat{q}, \hat{\theta}), \bar{\theta}(\bar{\theta}, s, \hat{q}, \hat{\theta})) \) and, recursively,\(^{33}\)
\[
\begin{align*}
\bar{c}_r^i(z^r) &= c^i(\theta^i_{r-1}(z_{r-1}), \hat{\theta}_{r-1}(z_{r-1}), z_r), \hat{q}, \hat{\theta}) = c^i(\hat{\theta}_{r-1}(z_{r-1}), z_r)
\bar{\theta}_r^i(z^r) &= \theta^i(\theta^i_{r-1}(z_{r-1}), \hat{\theta}_{r-1}(z_{r-1}), z_r), \hat{q}, \hat{\theta}) = \theta^i(\hat{\theta}_{r-1}(z_{r-1}), z_r).
\end{align*}
\]
Therefore, \( (\bar{c}_r^i(z^r), \bar{\theta}_r^i(z^r)) \in B^i(\hat{\theta}_{r-1}(z_{r-1}), z_r, \hat{q}(\hat{\theta}_{r-1}(z_{r-1}), z_r)) \) for all \( r \in \mathbb{N} \) by (4) and hence \( (\bar{c}_r^i, \bar{\theta}_r^i) \in F^i(\bar{\theta}, z, \hat{q}) \).

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\(^{32}\)See Stokey and Lucas Chapter 9 for more details about the composition of the stochastic kernels \( \lambda^i \).

\(^{33}\)This plan is measurable because the recursive equilibrium is measurable. Moreover, the feasibility follows directly from (4).
Replacing \((c^i, \theta^i)\) by \((\tilde{c}^i, \tilde{\theta}^i)\) in the previous arguments, we conclude that all inequalities must bind and hence \(v^i(\tilde{\theta}^i, s, \tilde{q}, \tilde{\theta}) = U^i(\tilde{c}^i, z) \leq \tilde{v}^i(\tilde{\theta}^i, z, \tilde{q})\) that is, \(v^i(\tilde{\theta}^i, s, \tilde{q}, \tilde{\theta}) = \tilde{v}^i(\tilde{\theta}^i, z, \tilde{q})\) and \((\tilde{c}^i, \tilde{\theta}^i) \in \delta^i(\tilde{\theta}^i, s, \tilde{q}).\)

Lemma 7.9. The correspondence \(\tilde{x}^i\) has a continuous selector.

Proof: Write \(N = \#Z + 1\). Consider \(h : \mathbb{R}^{HN} \times \mathbb{R} \to \mathbb{R}^N\) defined by \(h(q^n, \theta) = (q^n_1 \theta, ..., q^n_N \theta)\) and \(g : \mathbb{R}^{HN} \times \mathbb{R}^N \to \mathbb{R}^H\) defined by \(g(q^n, y) = \sum_{k \leq N} y_k q^n_k\). Clearly, the restriction of \(h(q^n, \cdot)\) to \(g(q^n, \mathbb{R}^N)\) is injective for each \(q^n \in \mathbb{R}^{HN}\) because \(h(q^n, \theta) = h(q^n, \theta')\) implies that \(34\) \(\theta - \theta' \in g(q^n, \mathbb{R}^N) \cap (g(q^n, \mathbb{R}^N))^\perp\). Moreover, the range of \(h(q^n, \cdot)\) and \(h(q^n, g(q^n, \mathbb{R}^N))\) coincide. Indeed, fix \(q^n \in \mathbb{R}^{HN}\) and suppose without loss of generality that \(\{q^n_n\}_{n \leq N}\) is the maximal linearly independent subset of \(\{q^n_n\}_{n \leq N}\) in \(\mathbb{R}^H\). Then, defining \(h' : g(q^n, \mathbb{R}^N \times \{0\}^{N-N}) \to \mathbb{R}^N\) by \(h'(\theta) = (q^n_1 \theta, ..., q^n_N \theta)\) then \(h'\) is bijective by the Dimension Theorem because \(h'\) is injective and linear. Therefore, given \(h(q^n, \theta)\) in the range of \(h(q^n, \cdot)\) then there exists \(\theta' := \sum_{k \leq N'} \alpha_k q^n_k\) such that \(q^n_\theta \cdot h'_n(\theta') = h_n(q^n, \theta) = q^n_\theta\) for all \(n \leq N'\) and hence for \(n > N'\) we get

\[
h_n(q^n, \theta') = q^n_\theta' = \sum_{k \leq N'} \alpha_k q^n_\theta' = \sum_{k \leq N'} \alpha_k q^n_\theta = q^n_\theta = h_n(q^n, \theta).
\]

Therefore \(h(q^n, \theta') = h(q^n, \theta)\) and \(\theta' \in g(q^n, \mathbb{R}^N)\) as desired. Define the correspondences \(\tilde{g}^i : S \times \mathring{Q}_\gamma \times \mathring{\Theta} \to \mathbb{R}^H\) and \(\tilde{h}^i : \Theta^i \times S \times \mathring{Q}_\gamma \times \mathring{\Theta} \to \mathbb{R}^H\) by

\[
\tilde{g}^i(s, \tilde{q}, \tilde{\theta}) = g(\tilde{q}^i(s), (\tilde{q}^i(\tilde{\theta}(s), z')) + \tilde{q}^c(\tilde{\theta}(s), z')d(z'))_{z' \in Z, R^N}
\]

and

\[
\tilde{h}^i(\theta^i, s, \tilde{q}, \tilde{\theta}) = h(\tilde{q}^i(s), (\tilde{q}^i(\tilde{\theta}(s), z') + \tilde{q}^c(\tilde{\theta}(s), z')d(z'))_{z' \in Z, \theta^i}).
\]

We claim that \(\tilde{x}^i(\theta^i, s, \tilde{q}, \tilde{\theta}) := \tilde{x}^i(\theta^i, s, \tilde{q}, \tilde{\theta}) \cap C^i \times \tilde{g}^i(s, \tilde{q}, \tilde{\theta})\) is a continuous function.\(^{35}\) Indeed, consider \((c^i, \theta^i), (c^i, \theta^i) \in \tilde{x}^i(\theta^i, s, \tilde{q}, \tilde{\theta})\) and let \((c^i, \theta^i)\) and

\(^{34}\)That is, the space orthogonal to \(g(q^n, \mathbb{R}^N)\).

\(^{35}\)Clearly, \(\tilde{x}^i\) is nonempty because given \((c^i, \theta^i) \in \tilde{x}^i(\theta^i, s, \tilde{q}, \tilde{\theta})\), there exists \(\tilde{\theta}^i \in \tilde{g}^i(s, \tilde{q}, \tilde{\theta})\) resulting in the same cost in the current period and in the same pay off at the next period. Interchanging \(\theta^i\) and \(\tilde{\theta}^i\) we can implement an optimum for the sequential demand, keeping the value of \(\tilde{\theta}^i\) unchanged. Since \(\tilde{v}^i = v^i\) by Theorem 7.8, then \(\tilde{\theta}^i \in \tilde{x}^i(\theta^i, s, \tilde{q}, \tilde{\theta}).\)
(\bar{c}^i, \bar{\theta}^i) be the optimal choices of the sequential problem choosing \theta^i and \bar{\theta}^i as the current optimal portfolio respectively. Then \bar{h}^i(\bar{\theta}^i, s, \hat{q}, \bar{\theta}) = \bar{h}^i(\bar{\theta}^i, s, \hat{q}, \bar{\theta}) because \bar{u}^i is strictly concave and strictly increasing.\textsuperscript{36} Otherwise we can implement, interchanging \theta^i and \bar{\theta}^i, a feasible stream of consumption providing higher expected utility, which is a contradiction. Therefore, \theta^i = \bar{\theta}^i since the restriction of \bar{h}(q^a, \cdot) to \bar{g}(q^a, \mathbb{R}^n) is injective for each \( q^a \in \mathbb{R}^{H_N} \).

To show the continuity, first note that \( \bar{x}^i \) is upper hemicontinuous by the Berge Maximum Theorem 17.31 in Aliprantis and Border (1999). Indeed, \( B^i \) is continuous by Lemma 7.2 as well as \( v^i \) and \( \bar{u}^i \). The continuity of \( \bar{x}^i \) follows from the fact that it is single valued and upper hemicontinuous since it is the intersection of an upper hemicontinuous and a closed graph correspondence.\textsuperscript{37} See Aliprantis and Border (1999) Theorem 17.27 for more details.

\[ \square \]

Lemma 7.10. For each compact \( \tilde{C} \times \tilde{\Theta} \subset \text{Int}(C \times \Theta) \) there exists \( \gamma > 0 \) such that if \( (\bar{\theta}, z, c, \theta, q) \) satisfies \( (c^i(\theta^i), \theta^i) \in \delta^i(\tilde{\theta}^i, z, q), (c_0^i, \theta_0^i) \in \tilde{C}^i \times \tilde{\Theta}^i \) and \( (c^i(z^r), \theta^i(z^r)) \in \tilde{C}^i \times \tilde{\Theta}^i \) for all \( z^r \in Z^r \) and all \( i \in I \) then \( q_0 \geq \gamma \).

Proof: Suppose, by a way of contradiction, that the statement does not hold. Then there exists \( \{s_n, c_n, \theta_n, q_n\}_{n \in \mathbb{N}} \) such that \( (c_n^i, \theta_n^i) \in \delta^i(\tilde{\theta}_n^i, z_n, q_n), (c_n^i(z^r), \theta_n^i(z^r)) \in \tilde{C}^i \times \tilde{\Theta}^i \) for all \( z^r \in Z^r \) and all \( i \in I \) and \( q_{0n} < 1/n \) for \( n \in \mathbb{N} \) and some\textsuperscript{38} \( l \in \mathcal{L} = \mathcal{J} \cup \mathcal{H} \). Using that \( Z \) is finite,\textsuperscript{39} we can suppose, passing to a subsequence if necessary, that \( \lim_{n \in \mathbb{N}} (s_n, c_n, \theta_n, q_n) = (\bar{\theta}, z, c, \theta, q) \) in the product topology. Since \( q_0 \in Q \), there exists \( k \in \mathcal{L} \) such that \( q_{0k} \neq 0 \). Moreover, \( \sum_i \theta^i_k = 1 \) allows to choose \( j \in I \) such that \( \bar{\theta}_k^j > 0 \). Using Proposition 2.10 we conclude that the value function \( \hat{v}^j \) given

\textsuperscript{36}This assures that there is only one optimal consumption choice and the budget inequality is binding. Thus \( \hat{\theta}(s) \theta^i = \bar{\theta}(s) \theta^i \).

\textsuperscript{37}To see that \( q^a \mapsto \bar{g}(q^a, \mathbb{R}^N) \) has closed graph, choose a convergent sequence in the graph of this map and use that \( \bar{g}(q^a, \mathbb{R}^N) \) can be generated by an orthonormal basis depending continuously on \( q^a \) by the Gram-Schmidt process. Indeed, a sequence given by a linear combination of a convergent orthonormal basis converges if and only if each parameter of the linear combination converges.

\textsuperscript{38}Note that we are using that \( L \) is finite.

\textsuperscript{39}The space of all measurable functions bounded by the same constant is compact under the topology of pointwise convergence since it is a closed subspace of the compact space of all functions from \( Z \) to \( \mathbb{R}^L \) bounded by the same constant. Indeed, a pointwise convergent net of measurable functions in this space is also measurable when \( Z \) is countable because the topology of pointwise convergence is second countable.
in Definition 2.8 is continuous on \((\bar{\theta}^j, z, q)\). Therefore, \((c^j, \theta^j) \in F^j(\bar{\theta}^j, z, q)\) and \(\hat{v}^j(\bar{\theta}^j, z, q) = U^j(z, c^j)\). But this is a contradiction because \(u^j\) is strictly increasing, \((c^j_0, \theta^j_0) \in \tilde{C}^j \times \tilde{\Theta}^j \subset \text{Int}(C^j \times \Theta^j)\) and \(q_{0l} = \lim_{n \in \mathbb{N}} q_{0ln} = 0^{40}\). □

References


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40\(c^j_0 \in \text{Int} C^j\) implies that the good prices must be positive for all \(r\) and hence asset prices at period one must be positive since dividends are strictly positive and agents can transfer consumption to next period increasing, in the current period, a small fraction of the asset at no additional cost.


