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Gustavo M. de Athayde, Renato Galvão Flôres Junior

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Finding a maximum skewness portfolio – a general solution to three-moments portfolio choice

Gustavo M. de Athayde^a, Renato G. Flôres Jr^{b,*}

^a*Banco Itaú, Rua Boa Vista 176, São Paulo 01014-919, Brazil*

^b*Escola de Pós-Graduação em Economia / Fundação Getulio Vargas, Praia de Botafogo 190, 11º andar, Rio de Janeiro 22253-900, Brazil*

Abstract

Considering the three first moments and allowing short sales, the efficient portfolios set for n risky assets and a riskless one is found, supposing that agents like odd moments and dislike even ones. Analytical formulas for the solution surface are obtained and important geometric properties provide insights on its shape in the three dimensional space defined by the moments. A special duality result is needed and proved. The methodology is general, comprising situations in which, for instance, the investor trades a negative skewness for a higher expected return. Computation of the optimum portfolio weights is feasible in most cases.

JEL classification: C49; C61; C63; G11

Keywords: Duality; Efficient set; Higher moments; Portfolio choice; Skewness.

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* Corresponding author. Tel.: + 55-21-2559 5909; fax: + 55-21-2553 8822.
E-mail address: rflores@fgv.br.

1. Introduction

Since at least Mandelbrot (1963)'s, economists and practitioners as well have been aware that the distribution of asset returns very seldom is normal. In spite of the considerable empirical literature now taking into account this fact (e.g. Campbell et al., 1997), financial theory has been reluctant in incorporating higher order moments in its developments. Portfolio choice – a most important subject for applications –, in spite of pioneer contributions like Samuelson (1967) and Kraus and Litzenberger (1976), still is largely dominated by the Markowitz *Weltanschauung*. In this paper we build up the efficient portfolios set for the case in which consideration is given to the three first moments. Ingersoll (1975), in an early effort in this direction, provided a clue on the shape of a portfolio frontier with the first three moments. Nevertheless his solution was based on differentiation of the utility function, and not on the technical constraints of the efficient set.

We provide the characteristics and the shape of the efficient set, as well as ways to assure the investor he is on it. Characterisation of the efficient set – i.e., the set of points where one can't get any better in any moment, without getting worse on another – seems a much more plausible and feasible approach if interest lies in actual computations related to practical problems as asset allocation, fund managing and dynamic portfolio optimisation. In practice, use of criteria based on utility functions may not seem reasonable for fund managers, especially those who need to report to their clients the criteria used to select the portfolios.

In general, investors will prefer high values for odd moments and low ones for even moments. The former can be seen as a way to decrease extreme values on the side of losses and increase them on the gains'. The latter can be justified by the fact that even moments measure dispersion, and therefore volatility; something undesirable because it increases the uncertainty of returns. If the investor is focussed on the worst scenario, like the *Value at Risk* of his portfolio, this kind of behaviour becomes even clearer and more justifiable.

Based on the assumption that agents like odd moments and dislike even ones, we shall build up the efficient portfolio frontier in the case of n risky assets and a riskless one, considering the three first moments and allowing short sales. The resulting surface will be represented in a convenient three dimensional space, whose co-ordinates are in the same units: excess expected returns, the cubic root of skewness and standard

deviations (the square root of variance). Given the assumption at stake, we shall in principle work in the positive orthant of $\mathbb{R}^3$ related to positive values in the three axes. There might however be cases where, for instance, the investor will be either obliged or open to trade a negative skewness for a higher expected return. Such situations are comprised in our methodology, and many of the results are presented in the half-space of $\mathbb{R}^3$, defined by the non-negativity constraint on the standard deviation.

The characterisation of the optimal surface will allow the actual computation of optimum weights in most practical cases. All results suppose the existence of the optimum, in the different optimisation programmes, what cannot be valid in certain pathological situations which are not considered in the paper.

It is interesting to see the optimal surface as the set of points giving the highest skewness for given mean and variance, but – as will be shown – they also give the highest mean for given variance and skewness, and the lowest variance for a given mean and skewness. In fact, it seems easier to attack the problem from this last viewpoint.

The next section develops a few preliminaries necessary for obtaining the results. They comprise a convenient notation and a duality proposition needed to pass from one “three-moments programme” to another. Section three then studies the efficient set for the minimum variance problem with given mean and skewness, while section four develops some insightful examples. With the help of the duality result, in section five we solve the complete problem. A final section concludes.

2. Preliminaries: notation and duality

2.1 A convenient notation

Dealing with higher moments can easily become algebraically cumbersome or even intractable. Given a n -dimensional random vector, its moments can be seen as tensors, a mathematical object well known to physicists, for instance. The second moments tensor is the popular $n \times n$ covariance matrix, while the third moments one can be visualised as a $n \times n \times n$ cube in three-dimensional space. As with the covariance matrix, filling up this whole cube amounts to include several identical values, as the

number of different skewnesses is equal to that of combinations with repetition of three

elements out of n , $\binom{n+2}{3}$, and not n^3 .

We shall transform the skewness tensor into a nxn^2 matrix by slicing each nxn layer and pasting them, in the same order, sideways. The fact that one ends up with a matrix allows the use of matrix differential calculus in all expressions and the derivation of compact and elegant formulas. Calling σ_{ijk} a general (co-) skewness, in the case that $n=2$, the resulting 2×4 matrix will be:

$$\begin{bmatrix} \sigma_{111} & \sigma_{112} & \sigma_{211} & \sigma_{212} \\ \sigma_{121} & \sigma_{122} & \sigma_{221} & \sigma_{222} \end{bmatrix},$$

of which only four elements are distinct.

Now suppose that a vector of weights $\alpha \in R^n$ is given, and x , M_2 and M_3 stand for the matrices containing the expected (excess) returns, co-variances and skewnesses of a random vector of n assets. The mean return, variance and skewness of the portfolio with these weights will be given, respectively, by: $\alpha'x$, $\alpha'M_2\alpha$ and $\alpha'M_3(\alpha \otimes \alpha)$, where ' $\otimes$ ' stands for the Kronecker product. It is immediate to see that, as real functions of α , these three expressions are homogenous functions of the same degree as the order of the corresponding moment. This means that Euler's theorem can be easily used in the needed derivations.

2.2 A duality result

Two kinds of duality results will be needed. The first is rather standard, relating a minimum of the objective function $f(x)$, constrained by one equality condition $\bar{g} - g(x) = 0$, with the maximum of the function in the condition, $g(x)$, now constrained by the objective function: $\bar{f} - f(x) = 0$ (or vice-versa). Theorem 9.12, in page 210, in Panik (1976), is suitable to our purposes, for instance. We remind only the basic requirements, namely that both f and g are of class C^2 in an open neighbourhood of the optimum and that the existence of a strong local minimum (or maximum) must be ensured.

However, we shall also need a duality result involving two equality constraints, something which is not very frequent in the standard optimisation literature. The lemma

below is especially customised to the needs of the next sections; ‘strict second order conditions’, in its hypotheses, means that all the signal inequalities of the bordered Hessians are strict (see the proof):

Duality Lemma: Let $f(x)$, $g(x)$ and $h(x)$, be real, continuously differentiable functions of class C^2 on an open set $A \subset \mathbb{R}^n$. If $x^* \in A$ is a strong (local) minimum of $f(x)$, subject to $\bar{g} - g(x) = 0$, and $\bar{h} - h(x) = 0$, $\bar{g}$ and $\bar{h}$ scalars, with corresponding Lagrange multiplier values given by λ_1 and λ_2 , $\lambda_1 > 0$, and strict second order conditions

THEN

$x^* \in A$ is also a strong (local) maximum of $g(x)$ subject to $f(x^*) - f(x) = 0$, and $\bar{h} - h(x) = 0$, with respective Lagrange multiplier values $1/\lambda_1$ and $-\lambda_2/\lambda_1$.

Proof: The proof is shown in the Appendix.

3. Minimum variance portfolios

The first problem we shall tackle is to minimise variance subject to skewness and mean return. Let M_1 , M_2 and M_3 stand for the matrices containing the mean returns, co-variances and skewnesses of n risky assets and r_f be the risk free rate of return. If $[1]$ is a $n \times 1$ vector of 1's, $\alpha \in \mathbb{R}^n$ is the vector of weights on the risky assets, the solution to the programme of finding a minimum variance portfolio with a given expected return $E(r_p)$ and skewness σ_{p^3} , is obtained by minimising the Lagrangian:

$$\text{Min}_{\alpha} L = \alpha' M_2 \alpha + \lambda_1 [E(r_p) - (\alpha' M_1 + (1 - \alpha' [1]) r_f)] + \lambda_2 (\sigma_{p^3} - \alpha' M (\alpha \otimes \alpha)) \quad (1)$$

Calling $x = M_1 - [1] r_f$, the vector of mean excess returns, and

$E(r_p) - r_f = R$, the given (excess) portfolio return,

the first order conditions are:

$$2M_2 \alpha = \lambda_1 x + 3\lambda_2 M_3 (\alpha \otimes \alpha) \quad (2)$$

$$R = \alpha' x \quad (3)$$

$$\sigma_{p^3} = \alpha' M_3 (\alpha \otimes \alpha) \quad (4)$$

Making use of the last two equations, we can find the values of λ_1 and λ_2 ,

$$\lambda_1 = \frac{A_4 R - A_2 \sigma_{p^3}}{A_0 A_4 - (A_2)^2} A_2, \quad \lambda_2 = \frac{A_0 \sigma_{p^3} - A_2 R}{A_0 A_4 - (A_2)^2} \quad (5)$$

$$\text{where: } A_0 = x' M_2^{-1} x, \quad ,$$

$$A_2 = x' M_2^{-1} M_3 (\alpha \otimes \alpha), \quad (6)$$

$$A_4 = (\alpha \otimes \alpha)' M_3' M_2^{-1} M_3 (\alpha \otimes \alpha);$$

the subscript of the A's corresponding to the degree of homogeneity of the term with respect to the vector α . Notice also that A_0 and A_4 are positive because the inverse of the covariance matrix is positive definite. Substituting (5) in (2) shows that the solution to the optimisation problem must satisfy the following n equations system:

$$M_2 \alpha = \frac{A_4 R - A_2 \sigma_{p^3}}{A_0 A_4 - (A_2)^2} x + \frac{A_0 \sigma_{p^3} - A_2 R}{A_0 A_4 - (A_2)^2} M_3 (\alpha \otimes \alpha) \quad (7)$$

System (7), which is highly non-linear in α , provides a necessary condition for the portfolios to be minimum variance. As there are two restrictions, a sufficient condition (see Panik, 1976, chapter 10) is that the determinants of the bordered Hessians below, for $r=3, 4, \dots, n$, be positive:

$$\begin{bmatrix} (2M_2 - 6 \frac{A_0 \sigma_{p^3} - A_2 R}{A_0 A_4 - (A_2)^2} M_3 (\alpha \otimes I))_{rr} & (x)_r & (M_3 (\alpha \otimes I))_r \\ (x)_r' & 0 & 0 \\ (M_3 (\alpha \otimes I))_r' & 0 & 0 \end{bmatrix}, \quad (8)$$

¹ Notice that λ_1 is the Lagrange multiplier of the first restriction, namely: $\bar{g} - g(x) = 0$..

where the symbol $(\)_{rr}$ denotes the square matrix obtained from the Hessian of (1) by retaining only the elements of its first rows and columns, while $(\)_r$ denotes the vector formed with the r first rows of the derivative of each restriction.

These two sets of conditions guarantee a local extreme, not necessarily a global one. There might be more than one solution to them and, in practice, due attempts must be made to ensure that the global minimum variance, subject to the constraints, was found. From now on, we shall suppose that there is at least one non-trivial solution to (6) satisfying the sufficient conditions in (8).

Pre-multiplying (6) by the solutions α' , the optimal variance(s) will be given by:

$$\sigma_{p^2} = \frac{A_4 R^2 - 2A_2 \sigma_{p^3} R + A_0 (\sigma_{p^3})^2}{A_0 A_4 - (A_2)^2} . \quad (9)$$

As the inverse of the covariance matrix is positive definite, both the numerator and denominator of the above expression are positive:

$$\begin{aligned} \{M_3(\alpha \otimes \alpha)R - x\sigma_{p^3}\}' M_2^{-1} \{M_3(\alpha \otimes \alpha)R - x\sigma_{p^3}\} &= A_4 R^2 - 2A_2 R \sigma_{p^3} + A_0 (\sigma_{p^3})^2 > 0 \\ \{M_3(\alpha \otimes \alpha)A_0 - xA_2\}' M_2^{-1} \{M_3(\alpha \otimes \alpha)A_0 - xA_2\} &= A_0 [A_0 A_4 - (A_2)^2] > 0 \end{aligned} .$$

Moreover, as $A_0 > 0$, the result in the second expression above allows to derive a non-negativity condition for the multipliers in (5):

$$\lambda_1 > 0 \Leftrightarrow A_4 R > A_2 \sigma_{p^3} \quad \text{and} \quad \lambda_2 > 0 \Leftrightarrow A_0 \sigma_{p^3} > A_2 R . \quad (10)$$

Without loss of generality, let us now fix a positive real number k , and consider all (σ_{p^3}, R) pairs such that $\sigma_{p^3} = k^3 R^3$, or $y_3 = kR$, where y_3 is the cubic root of skewness, also called standardised skewness. The following important Proposition is true:

Proposition 1: For a given k , let $\bar{\alpha}$ define the minimum variance portfolio when $R=1$ and $y_3 = k$, and $\bar{\sigma}_{p^2}$, $\bar{\lambda}_1$, $\bar{\lambda}_2$ be, respectively, the corresponding minimum variance and multipliers, THEN

for all optimal portfolios related to skewness/return pairs such that $\sigma_{p^3} = k^3 R^3$, or $y_3 = kR$:

- i) the solution to (7) will be $\alpha = \bar{\alpha}R$, with corresponding minimum variance $\sigma_{p^2} = \bar{\sigma}_{p^2} R^2$;
- ii) the respective Lagrange multipliers (5) will be related by:
 $\lambda_1 = \bar{\lambda}_1 R$ and $\lambda_2 = \bar{\lambda}_2 R$;
- iii) the two funds separation property is valid.

Proof: Rewriting (7), when $\sigma_{p^3} = k^3 R^3$, or $y_3 = kR$:

$$M_2 \alpha = \frac{A_4 R - A_2 k^3 R^3}{A_0 A_4 - (A_2)^2} x + \frac{A_0 k^3 R^3 - A_2 R}{A_0 A_4 - (A_2)^2} M_3 (\alpha \otimes \alpha) . \quad (11)$$

When $R = 1$, calling $\bar{\alpha}$ the solution, this reduces to:

$$M_2 \bar{\alpha} = \frac{\bar{A}_4 - \bar{A}_2 k^3}{A_0 \bar{A}_4 - (\bar{A}_2)^2} x + \frac{A_0 k^3 - \bar{A}_2}{A_0 \bar{A}_4 - (\bar{A}_2)^2} M_3 (\bar{\alpha} \otimes \bar{\alpha}) . \quad (12)$$

It is easy to see that $\alpha = \bar{\alpha}R$ solves (11); indeed, substituting it in the equation and remembering that the A 's are homogenous functions gives:

$$M_2 \bar{\alpha} R = \frac{\bar{A}_4 R^5 - \bar{A}_2 k^3 R^5}{[A_0 \bar{A}_4 - (\bar{A}_2)^2] R^4} x + \frac{A_0 k^3 R^3 - \bar{A}_2 R^3}{[A_0 \bar{A}_4 - (\bar{A}_2)^2] R^4} M_3 (\bar{\alpha} \otimes \bar{\alpha}) R^2 ,$$

after cancelling the R 's, one is back to (12). The result for the variance is straightforward.

The multipliers relation is also straightforward. Taking λ_1 for instance:

$$\lambda_1 = \frac{A_4 R - A_2 \sigma_{p^3}}{A_0 A_4 - (A_2)^2} A_2 = \frac{\bar{A}_4 R^5 - \bar{A}_2 \bar{\sigma}_{p^3} R^5}{A_0 \bar{A}_4 R^4 - (\bar{A}_2)^2 R^4} \bar{A}_2 R^2 = \bar{\lambda}_1 R \quad .$$

Now, all minimum variance portfolios that are in a fixed direction k in $R \times y_3$ space are simply multiples of $\bar{\alpha}$. Along each of these lines, the only thing that changes among the optimal portfolios is how much is allocated on the “risky portfolio $\bar{\alpha}$ ” and then, on the riskless asset, to satisfy the unit sum condition. Thus, for every angle k , a two fund separation property is guaranteed.

In the direction $y_3 = kR$, the optimal variance as a function of the excess return will be a parabola. Taking instead the three dimensional space defined by standard deviation $\sqrt{\sigma_{p^2}} = y_2$, standardised skewness and (excess) return, in the half-plane formed by a specific direction k (in $R \times y_3$ space) and the (positive part) of the standard deviation axis, the optimal portfolio surface will be reduced to the straight line $y_{p^2} = \bar{y}_{p^2} \frac{x}{\sqrt{k^2 + 1}}$, $x \geq 0$ ². As $\bar{y}_{p^2}$ differs with k , the angle that this line makes with the standard deviation axis varies also with k , which, in turn, also specifies an angle/direction in the first quadrant of the $R \times y_3$ plane. Figure 1 gives an idea of the situation.

Figure 1: (a) the optimal variance parabola in mean (excess returns) $\times$ standardised skewness $\times$ variance space; (b) the corresponding straight line in mean $\times$ standardised skewness $\times$ standard deviation space

² The variable x stands for the coordinates along the axis defined by the “direction k ”.

One may ask what will be the shape of the whole set when the k directions vary. In principle, four types of surfaces, roughly sketched in Figure 2, can be conceived:

Figure 2: Possible shapes of the optimal three moments set in (excess) mean returns x standardised skewness x standard deviation space.

The areas in light grey represent the surface formed by the set of minimum variance lines/portfolios outlined in Figure 1(b). In Figure 2(d) we have some directions that give us either local maxima R or local maxima y_3 , for a given level of y_2 . In Figure 2(c) there are several maxima R and only one direction giving a maximum y_3 , while in 2(b) there are several local maxima y_3 and only one maximum R . Finally, 2(a) represents a sort of ideal situation where there is only one maximum R and one maximum y_3 for a given y_2 . It will be shown that the cases described in 2(c) will never happen, and so those in 2(d).

Proposition 1 and the pictures outlined in Figures 1 and 2 have a far reaching consequence. The optimal surface in the positive-variance half of 3D space bears a homothetic property from whatever standpoint one assumes, i.e., slicing the surface by a

sequence of planes parallel to any two axes will generate a sequence of homothetic figures starting at the origin and whose expansion ratio will be equal to that of the respective (excess) returns. In particular, the following Proposition is immediate:

Proposition 2: For a given level of y_2 , cut the optimal surface with a plane orthogonal to the standard deviations axis and project the intersection in the ‘returns x skewness plane’, THEN the k directions (in the $R \times y_3$ half plane) related to the highest R and the highest (and lowest) y_3 are invariant with y_2 .

Henceforth, if we want to see the set of minimum variance portfolios that have the highest R or y_3 , it suffices to find such a portfolio in a given level of variance. The desired set will be made of multiples of these portfolios. But how do we find these specific directions? The answer is in the following proposition:

Proposition 3: The direction in the $R \times y_3$ half plane that gives the highest R for all the minimum variance portfolios with the same standard deviation y_2 is unique and related to the celebrated (Markowitz’s) Capital Market Line (CML). Moreover, in this direction, the skewness constraint to programme (1) is not binding. A similar property holds for the directions giving the highest (and lowest) skewness, but the directions are now not unique.

Proof: Notice that, in view of the classical duality result, the highest R for a given variance level σ^2 , when fixed, produces a minimum variance that equals the very σ^2 . We shall then begin by considering the classical mean-variance problem:

$$\text{Min } \alpha' M_2 \alpha + \lambda (R - \alpha' x) \quad , \text{ with notation as before.}$$

The first order conditions are:

$$2M_2 \alpha = \lambda x \quad \text{and} \quad R = \alpha' x \quad .$$

Pre-multiplying the first condition by $x' M_2^{-1}$, we have: $\lambda = \frac{2R}{A_0}$. Substitution

of this value in the second one yields the unique solution:

$$\alpha = \frac{R}{A_0} M_2^{-1} x \quad . \quad (13)$$

Calling this portfolio α_R , the associated optimal variance, solution to this classical problem, is

$$\sigma_{R^2} = \frac{R^2}{A_0} \quad , \quad (14)$$

defining the famous *Capital Market Line* in mean x variance space. A skewness and a k can also be associated to this optimal portfolio, being evident that both are independent of the given y_2 and of the solution α_R , implying that – as expected – one also has a straight line in mean x skewness space:

$$\sigma_{R^3} = \frac{R^3}{A_0^3} w' M_3 (w \otimes w) \quad , \quad \text{where } w = M_2^{-1} x \quad , \text{ and} \quad (15)$$

$$k_R = \frac{y_{R^3}}{R} = \frac{\sqrt[3]{w' M_3 (w \otimes w)}}{A_0} \quad . \quad (16)$$

Hence, the direction k_R is an invariant, and, by duality, all the maximum mean returns (for given variance) lie in the same direction in mean x skewness space.

It is also easy to verify that (13) can be a solution to (7), given R and the implied skewness. It suffices to notice that:

$$A_{2_R} = x' M_2^{-1} M_3 (\alpha_R \otimes \alpha_R) = \frac{R^2}{A_0^2} w' M_3 (w \otimes w) \quad \sigma_{R^3} = \frac{R}{A_0} A_{2_R} \quad , \quad (17)$$

which, when applied to (7), provides the required answer. Moreover, combining (17) and (5) implies that $\lambda_2 = 0$, making the skewness constraint not binding and redundant all along direction k_R . So, all the results are proved.

For the case of skewness, following similar steps, we first find the portfolios that minimise variance subject only to skewness, i.e.,

$$\text{Min } \alpha' M_2 \alpha + \lambda (\sigma_{p^3} - \alpha' M_3 (\alpha \otimes \alpha)) \quad .$$

These portfolios will be called α_s . The first order conditions are:

$$2M_2\alpha_s = \lambda 3M_3(\alpha_s \otimes \alpha_s) \quad \text{and} \quad \sigma_{s^3} = \alpha_s' M_3(\alpha_s \otimes \alpha_s) \quad .$$

Pre-multiplying the first one by $(\alpha_s \otimes \alpha_s)' M_3' M_2^{-1}$, we have: $\lambda = \frac{2\sigma_{s^3}}{3A_{4_s}}$,

which entails:

$$\alpha_s = \frac{\sigma_{s^3}}{A_{4_s}} M_2^{-1} M_3(\alpha_s \otimes \alpha_s) \quad . \quad (18)$$

Contrary to the previous (classical) case, the optimal weights are now (as usual) implicitly defined by a non-linear system. When $\sigma_{s^3} = 1$, we have a portfolio $\bar{\alpha}_s$ such that:

$$\bar{\alpha}_s = \frac{1}{A_{4_s}} M_2^{-1} M_3(\bar{\alpha}_s \otimes \bar{\alpha}_s) \quad . \quad (19)$$

The homotethy implies that $\alpha_s = \bar{\alpha}_s \sqrt{\sigma_{s^3}} = \bar{\alpha}_s y_{s^3}$ is a solution to (18), ensuing an optimal variance $\sigma_{s^2} = \bar{\sigma}_{s^2} (y_{s^3})^2$. A corresponding (excess) return and a direction, both independent of the variance level, can be found as:

$$R_s = \bar{R}_s y_{s^3} \quad , \quad k_s = 1/\bar{R}_s \quad , \quad (20)$$

implying that all these optimal portfolios lie in the same direction. To verify that it also is a particular case of (9) in the direction k_s , pre-multiply (18) by x' , resulting in:

$$R_s = \frac{\sigma_{s^3}}{A_{4_s}} A_{2_s} \quad . \quad (21)$$

Supposing $\sigma_{s^3} = 1$, use of (21) in (9) will give:

$$\bar{\sigma}_{s^2} = \frac{1}{A_{4_s}} \quad , \quad (22)$$

which is exactly what one gets when, using the (implicit) expression for $\bar{\alpha}_s$ in (19), one computes the minimum variance. Moreover, (21) makes the first constraint of (1) – shown in (5) – redundant, implying that it is a special case of the general problem, when $\lambda_1 = 0$. Thus, again by duality, the result is proved, and, in this situation, all we need is

to find the portfolio that minimises variance subject to the constraint that skewness is equal to 1.

Notice that, in the ‘Markowitz case’, in order to achieve the requirements of classical duality – i.e., that (14) is indeed a strong local minimum – nothing is demanded of the sign of λ_1 , as the multiplier disappears in the second derivative. However, it will be important to know whether $\lambda_1 > 0$. Working with the corresponding formula in (5), this implies that:

$$\lambda_1 = \frac{A_4 R - A_2 \sigma^3}{A_0 A_4 - (A_2)^2} > 0 \quad \frac{A_4 R - (A_2)^2 R / A_0}{A_0 A_4 - (A_2)^2} = \frac{R}{A_0} > 0 \quad R > 0 \quad . \quad (23)$$

So, once the direction is found, if one works with portfolios that have a positive R , the multiplier will be also positive. It should be emphasised that there is a unique direction in this case, described by (16). Thus, the shapes sketched in Figures 2c and 2d will never occur.

In the skewness case, the multiplier must be positive for duality to be ensured. Using (21) in the condition for $\lambda_2 > 0$ in (10), yields:

$$A_0 \sigma_{p^3} > (A_2)^2 \frac{\sigma_{p^3}}{A_4} \quad ,$$

so that, if the skewness is positive, by the derivations following formula (9), the condition is satisfied. However, given that (19) can be a non-linear system of, in principle, the fifth degree, there will usually be more than one solution to it; and consequently more than one k_s .

4. Examples and further insights

We begin by finding the ‘highest skewness line’ when all the covariances and coskewnesses between n different risky assets are null. This is interesting first because in the standard Markowitz world the solution is always an interior point, what will never happen now. Secondly, as will be seen, even in a simpler structure like this there can exist a considerable multiplicity of solutions to (19).

Working out system (19), it is not very hard to see that one arrives at the following set of equations:

$$\bar{\alpha}_i^2 \frac{\sigma_{iii}}{\sigma_{ii}} = \alpha_i \prod_{j=1}^n \frac{\sigma_{jj}^2}{\sigma_{jj}} \bar{\alpha}_j^4, \quad 1 \leq i \leq n. \quad (24)$$

The above system, beyond the trivial solution at the origin, has $2^n - 1$ non-trivial ones obtained by setting to zero a subset of weights – here included the null set. Calling K the set of non-zero weights, the solution for one $i \in K$, with the aid of the general first order conditions used in the proof of Proposition 3, will be:

$$\bar{\alpha}_i = \frac{\sigma_{ii}}{\sigma_{iii}} \left/ \left(\prod_{j \in K} \frac{\sigma_{jj}^3}{\sigma_{jj}^2} \right)^{1/3} \right. . \quad (25)$$

This means that there is only one solution with all weights different from zero and $2^n - 2$ corner solutions where at least one asset does not enter in the optimal portfolio. The variance associated to a set K solution will be:

$$\left(\prod_{j \in K} \frac{\sigma_{jj}^3}{\sigma_{jj}^2} \right)^{1/3} .$$

It is then evident that the global minimum will correspond to a corner solution in which all weights are zero but the one which will assign $(\sigma_{iii})^{-1/3}$ to the asset i^* for which the quantity

$$\sigma_{ii} / \sigma_{iii}^{2/3} \quad (26)$$

attains its smallest value. All the other solutions will be related to local minima or maxima (it is not difficult to prove that the non-zero-weights one corresponds to a maximum). The direction of interest – out of the $2^n - 1$ possible ones, it is important to remind – will be defined by an angular coefficient

$$k_S = (\sigma_{i^*i^*i^*})^{1/3} / (r_{i^*} - r_f),$$

where r_{i^*} is the mean return of asset i^* .

It is intuitive that the extreme corner solutions yield the best results. Indeed, as said in the proof of Proposition 3, pre-multiplying (19) by αM_2 one gets (22):

$\bar{\sigma}_{ss} = \frac{1}{\bar{A}_4}$. Therefore, the highest $\bar{A}_4$ will produce the lowest variances. When all the co-variances and coskewnesses are null,

$$\bar{A}_{4s} = \sum_{i=1}^n \frac{\alpha_i^4 \sigma_{iii}^2}{\sigma_{ii}} ;$$

since this is a convex function on the α 's, the corner solutions will give the highest $\bar{A}_4$. Thus, concentrating on any single asset will bring forth optimal solutions. In this case, the amount to be invested on the single asset will have to obey the following constraint:

$$1 = \alpha_i^3 \sigma_{iii} \quad \alpha_i = \frac{1}{\sqrt[3]{\sigma_{iii}}} .$$

To achieve the highest $\bar{A}_{4s}$, one should invest everything on the asset that has the highest:

$$\frac{\alpha_i^4 \sigma_{iii}^2}{\sigma_{ii}} = \frac{(\sigma_{iii})^{\frac{2}{3}}}{\sigma_{ii}} . \quad (27)$$

The intuition of choosing such portfolio is quite simple: it is the one which, keeping the predetermined skewness level (in this case, 1), increases to the least possible the total variance.

This example is interesting because while the CML usually gives an interior solution, in which we diversify our portfolio, the directions with the highest skewness are those that are totally concentrated on a single asset. In practice, when the hypothesis of null co-variances and co-skewnesses is not observed, the criterion based on (27) could be a *warm start* for the iterations needed to solve system (7).

Moving back to the general case, it is important to find the direction in which there will be a global minimum variance for a given skewness (or maximum skewness for a given variance). However, as shown by the previous example, there will be other (local) minimum variance directions, as well as (global and local) maximum variance ones. Figure 3 shows four examples, computed with simulated data, of how the isovariance curve may look like in the “returns x skewness” plane. All the four curves have two directions that provide, locally, the highest skewness (in absolute values) and one in which the curve bends, giving the lowest skewness (in absolute values). When

one of the first two directions becomes very close to the latter, the curve becomes closer to an elliptical shape, as in the two bottom pictures. It is also worth noticing that the direction that gives a highest (lowest) skewness in one quadrant gives a lowest (highest) one in the diagonally opposed one.

If the portfolio with the highest skewness is the one further to the right – i.e., generating points in the first quadrant, we'll have a well-behaved efficient set. If it is further to the left - lying, for instance, in the second quadrant, we'll have a discontinuous efficient set, or, if one only wants to work with non-negative skewness and excess return, the surface must be cut.

Figure 3: Four possible shapes of isovariance curves in the mean x standardised skewness plane. The curves were generated from actual data points, that is why they look discontinuous. Beyond being all continuous, the sharp horizontal edges suggested in all but the one below right do not really characterise cusps, all being (if conveniently zoomed) related to smooth extrema.

Figure 4 – a qualitative interpretation of several simulations - illustrates two key cases. The one to the left corresponds to the ideal situation, the part of interest in the isovariance curves lies in the first quadrant and is a connected set. In the one to the right – a mirror image of the previous curve -, the global maximum skewness corresponds to a negative return value. Moreover, the valley of the lowest skewness solution forces a discontinuity in the optimal set. If the discontinuity is very long, it will imply a great loss of expected return when jumping from one part to the other of the efficient set, meaning that we might be better off staying at the right side arch. As a consequence, the optimal portfolio set may assume many different shapes beyond the well-behaved paraboloid-like surface in Figure 2(a). In practice, when facing a situation like the one in the right of Figure 4, the analyst must decide where she wants to pursue the minimisation algorithm: work in both parts, stay only with the right arch or work with it and the portion of the other arch in the first quadrant.

Figure 4: Two possible shapes of isovariance curves, with – in bold – the line from the maximum mean return point to the maximum skewness one. The figure in the left portrays the ideal situation, in the one in the right, the final efficient surface will have two pieces.

As a final remark on why the isovariance curves may become so complicated, it should perhaps be reminded that portfolio skewness can become a very complicated function of R , depending on the (marginal) skewnesses and co-skewnesses between the different assets. Even with a small number of assets, say three, it is easy to work out a formula like (9) and see that it will give origin to an irrational polynomial in R , of

degree never smaller than three. The “isovariance curves” may then assume different and complex shapes as illustrated above

5. The efficient frontier

The efficient frontier will be built up by means of the Duality Lemma; in order to conveniently use it a last proposition is needed:

Proposition 4: Let k_R be as in (14) and the k_S in (20) be the one related to the highest skewness line, which is supposed to be unique, THEN $k_S \geq k_R$, or, in other words, the angle of the ‘highest skewness line’ is greater than the one of the ‘highest mean (excess) return line’. Moreover, in the area comprised by these two lines and the positive (excess) returns half of the mean returns x standardised skewness plane, both Lagrange multipliers related to programme (1) are positive.

Proof: Let $R=1$ and consider the (irrestrict) minimum variance (14) which solves the classical mean-variance problem; there is associated to this pair a standardised skewness obtained from (15) which is exactly k_R . Now take in the ‘highest skewness line’ the (standardised) skewness corresponding to $R=1$, which will be k_S . The variance associated to the constraints 1, k_S must be superior to (14), because this is the (skewness unconstrained) minimum variance when $R=1$. But it is also the (mean unconstrained) minimum variance when skewness equals k_S , so the only possibility is that $k_S \geq k_R$.

For the second part, we know that λ_2 is zero all along the ‘highest mean return line’ and, from the previous argument, that the ‘highest skewness line’ lies above. Given that the multipliers are a continuous function of the constraints, let’s examine what happens with λ_2 when we move upwards, in the vertical, from a point on the ‘highest mean return line’. Along this path, the mean return will be the same but skewness will be progressively increasing; the fact that departure was from a “Markowitz point” and the homotethy imply that the minimum variance will also be increasing, so λ_2 will become positive – the optimum is increasing with the restriction - along all lines progressively passing through the origin at a higher angle than k_R . This continues until one reaches the ‘highest skewness line’ where λ_2 is indeed positive whenever the skewness is, as shown at the end of section 3.

For λ_1 it suffices to repeat the argument now moving horizontally to the right, from a point on the ‘highest skewness line’ - where, as known, λ_1 is zero. The value of the multiplier when one reaches the other line is, from (2), positive whenever R is.

We are now ready to invoke the duality result. Within the region between the two canonical lines both multipliers are positive and the Lemma applies to both possible inversions, namely, maximise skewness, given the same mean and the optimum variance, or maximise mean excess return given the same skewness and the optimum variance, so that the efficient frontier is the part of the minimum variance surface (outlined in the ‘normal case’ in Figure 2(a)) comprised between the vertical planes passing through the canonical lines. Outside this part, though for a given mean-skewness pair there exists a minimum variance, as at least one multiplier is negative, it will be possible to move a little in the optimal surface, along a specified direction, increasing both members of the pair while simultaneously decreasing the variance; so that the original pair gives way to an inefficient point.

For computing the optimum, one should then first obtain the canonical lines described in Proposition 3; vertical – i.e., parallel to the standard deviation axis – planes through them will circumscribe the efficient set. If a minimum variance portfolio is sought, provided the k defined by the ratio of the given skewness to the cube of the given excess return is efficient – i.e., comprised between the two canonical lines the agent wants to work³ -, the fixed point(s) for the function φ below, which is a re-working of the right-hand side of (11), making $\sigma_{p^3} = k^3$ and $R = 1$,

$$\varphi(\bar{\alpha}) = \frac{A_4(\bar{\alpha}) - A_2(\bar{\alpha})k^3}{A_0A_4(\bar{\alpha}) - (A_2(\bar{\alpha}))^2} M_2^{-1}x + \frac{A_0k^3 - A_2(\bar{\alpha})}{A_0A_4(\bar{\alpha}) - (A_2(\bar{\alpha}))^2} M_2^{-1}M_3(\bar{\alpha} \otimes \bar{\alpha}) \quad ,$$

will be the set of weights $\bar{\alpha}$. Multiplication of this vector by the given mean excess return will produce the solution.

6. Conclusions

We have shown how to deal, in a general way, with three-moments portfolio choice. For reasons we think mostly related to the algebraic intractability of the

problem, no complete solution to it had been presented till now, in spite of the nowadays common knowledge that skewness matters. The notation used allowed to derive compact formulas ready to feed the proper optimisation algorithms. Moreover, a deep insight on the geometry of the efficient set was obtained, which – beyond its theoretical interest – is crucial to guide the practical finding of the optimal weights.

The results presented should now be used in several and diversified concrete situations, to produce a better grasp of when the (simpler) Markowitz solution would lie too far from the optimum and to improve the knowledge on the different patterns of multiple local optima. The example carried out in section four, and the complex structure – in its full generality - of the nonlinear system (7) make these practical experiments a must. They also seem to announce that the Markowitz solution may be very inefficient.

Finally, given the need to regularly update optimal portfolios, the framework here developed must be translated, in a further stage, into a dynamic setting.

Appendix

Proof of the Duality Lemma in section 2:

An extreme value of $f(x)$, at point x^* , subject to $\bar{g} - g(x) = 0$, and $\bar{h} - h(x) = 0$ must satisfy:

$$D_f - \lambda_1 D_g - \lambda_2 D_h = 0 \quad , \quad \text{at } x^* \quad . \quad (\text{A.1})$$

As $\lambda_1 > 0$, multiplying (A.1) by $-1/\lambda_1$, we get:

$$D_g - \gamma_1 D_f - \gamma_2 D_h = 0 \quad , \quad \text{at } x^* \quad , \quad (\text{A.2})$$

where $\gamma_1 = 1/\lambda_1 > 0$ and $\gamma_2 = -\lambda_2/\lambda_1$. (\text{A.3})

Given that x^* is a minimum of $f(x)$ subject to $\bar{g} - g(x) = 0$ and $\bar{h} - h(x) = 0$, the differentiability hypotheses imply that the determinants of the Bordered Hessians below – where $r=3, 4, \dots, n$, and the symbol $(\)_{rr}$ denotes the square matrix obtained from the Hessian of the Lagrangian by retaining only the elements of its first r rows and columns, while $(\)_r$ denotes the vector formed with the r first rows of the derivative of each restriction – exist. Moreover, as strict second-order conditions are assumed, by Theorem 10.5 in Panik (1976; page 220), they must also be non-negative,

³ See figure 4 for more than one possibility.

$$\begin{bmatrix} (H_f - \lambda_1 H_g - \lambda_2 H_h)_{rr} & (-D_h)_r & (-D_g)_r \\ (-D_h)_r' & 0 & 0 \\ (-D_g)_r' & 0 & 0 \end{bmatrix} . \quad (\text{A.4})$$

Multiplying the matrix in (A.4) by $-1/\lambda_1$, we have:

$$\begin{bmatrix} (H_g - \gamma_1 H_f - \gamma_2 H_h)_{rr} & (D_h)_r / \lambda_1 & (D_g)_r / \lambda_1 \\ (D_h)_r' / \lambda_1 & 0 & 0 \\ (D_g)_r' / \lambda_1 & 0 & 0 \end{bmatrix} , \text{ where now the sign of each}$$

determinant will be that of $(-\lambda_1)^{r+2}$ or of $(-1)^{r+2}$ as $\lambda_1 > 0$. Multiplying the last two rows and columns of each matrix above by λ_1 :

$$\begin{bmatrix} (H_g - \gamma_1 H_f - \gamma_2 H_h)_{rr} & (D_h)_r & (D_g)_r \\ (D_h)_r' & 0 & 0 \\ (D_g)_r' & 0 & 0 \end{bmatrix} ,$$

and the determinant signs will not change. Making use of (A.2), these matrices are identical to:

$$\begin{bmatrix} (H_g - \gamma_1 H_f - \gamma_2 H_h)_{rr} & (D_h)_r & (\gamma_1 D_f + \gamma_2 D_h)_r \\ (D_h)_r' & 0 & 0 \\ (\gamma_1 D_f + \gamma_2 D_h)_r' & 0 & 0 \end{bmatrix} ;$$

moreover, their determinants do not change if one subtracts the penultimate row times γ_2 from the last, and does the same for the last two columns:

$$\begin{bmatrix} (H_g - \gamma_1 H_f - \gamma_2 H_h)_{rr} & (D_h)_r & (\gamma_1 D_f)_r \\ (D_h)_r' & 0 & 0 \\ (\gamma_1 D_f)_r' & 0 & 0 \end{bmatrix} .$$

Multiplying now the penultimate row and column by -1 and the last ones by

$$-1/\gamma_1 = \lambda_1 > 0 :$$

$$\begin{bmatrix} (H_g - \gamma_1 H_f - \gamma_2 H_h)_{rr} & (-D_h)_r & (-D_f)_r \\ (-D_h)_r' & 0 & 0 \\ (-D_f)_r' & 0 & 0 \end{bmatrix} . \quad (A.5)$$

But the matrices in (A.5) are Bordered Hessians, for $r=3,\dots,n$, of the problem of maximising $g(x)$ subject to $\bar{f} - f(x) = 0$ and $\bar{h} - h(x) = 0$. Their signs, given the last operation performed, will be those of $(-1)^{r+4}$. Equation (A.2) and the fact that $(-1)^r$ times the corresponding determinant in (A.5) always gives a positive number (as $(-1)^r (-1)^{r+4} > 0$) make up a sufficient condition (by the already mentioned Theorem 10.5 in Panik(1976)) for x^* to be a strong local maximum of the dual problem.

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