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On the Efficiency of Equal Sacrifice Income Tax Schedules*

Carlos E. da Costa, Thiago Pereira

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On the Efficiency of Equal Sacrifice Income Tax Schedules*

Carlos E. da Costa

FGV-EPGE

carlos.eugenio@fgv.br

Thiago Pereira

FGV-EPGE

tpereira@fgvmail.br

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Abstract

In an economy which primitives are exactly those in Mirrlees (1971), we investigate the efficiency of labor income tax schedules derived under the equal sacrifice principle. Starting from a given government revenue level, we use Werning's (2007b) approach to assess whether there is an alternative tax schedule to the one derived under the equal sacrifice principle that raises more revenue while delivering less utility to no one. For our preferred parametrizations of the problem we find that inefficiency only arises at very high levels of income. We also show how the multipliers of the Pareto problem may be extracted from the data and used to find the implicit marginal social weights associated with each level of income. *Keywords:* Equal Sacrifice; Efficiency. *J.E.L. codes:* H2; D63.

1 Introduction

Mirrlees's (1971) has defined the standard for normative income taxation: the maximization of a social welfare functional subject to incentive constraints. Despite its indisputable methodological advantages, the consensus regarding this procedure has somehow overshadowed the discussion of the principles of distributive justice that justifies the use of a social welfare function in the first place. This is important since Welfarism is but one of the possible views of distributive justice that one may adopt. It need not necessarily capture a society's view of what a 'just' tax is.

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Renewed interest in alternative views of justice, and their consequences for optimal taxation is reflected by the emergence of a new wave of works aimed at, on the one hand, better understanding the society's view of what a just tax system is, e.g. Weinzierl (2012a), and, on the other, using this information to guide policy, e.g., Weinzierl (2012b); Saez and Stantcheva (2013).

In this paper we revisit one such non-welfarist criteria, which played an important role in the debate of distributive justice throughout the nineteenth and most of the twentieth century: the Equal Sacrifice principle. This principle is aptly explained by John Stewart Mill as follows "...whatever sacrifices the government requires should be made to bear as nearly as possible with the same pressure upon all"—see Mill (1844).¹

Our focus on the equal sacrifice principle is motivated by two separate, but related findings. First, are recent survey results that suggest that Equal Sacrifice is still held by many as the basis of what should guide policy. More precisely, Weinzierl (2012a) finds that three fifths of survey respondents prefer the use of the Equal Sacrifice principle to a Utilitarian metric to guide policy. Second is the fact that in a series of papers in the late 1980's Young (1987, 1988, 1990) conclude that most tax schedules that prevailed in the US for the period 1957 to 1987 may be rationalized by direct applications of the equal sacrifice principle.² This suggests that, not only is the Equal sacrifice idea still pervasive, but that it may have actually influenced policy design.

The question we pose in this paper is whether equal sacrifice schedules are efficient. There are several reasons why we may be interested in efficiency. Most important of all is the fact that many ideals of fairness yield to the notion of efficiency, sometimes through a direct axiomatic imposition of efficiency, and often as an ad-hoc deviation from the pure application of a fairness ideal. ³This possibility, along with the apparent empirical relevance of the equal sacrifice principle motivates the assessment of efficiency of equal sacrifice schedules, the focus of our work.

A characteristic of the early literature on equal sacrifice precludes us from relying on it for assessing efficiency: it (implicitly) takes taxable income to be independent of the tax schedule. Because no behavioral responses take place, inefficiency may not arise.⁴

¹Another view of justice that violates welfarism, the *benefit doctrine*, seems to still be playing a role in political discourse. Indeed, this is the underlying principle adopted by President Barack Obama in his call for the wealthy to pay more taxes on the grounds that others have "helped to create this unbelievable American system that we have that allowed you to thrive". See also footnote 7.

²The body of work that followed allowed us to better understand not only the restrictions imposed on observed tax schedules by the equal sacrifice system — Mitra and Ok (1996); Ok (1995) — but also the consequences of taking incentives into account explicitly — Berliant and Gouveia (1993).

³While exploring different notions of fairness, for example, both Weinzierl (2012b) and Saez and Stantcheva (2013) impose Pareto efficiency as a 'sensible property' to be satisfied by the resulting allocations.

⁴Young (1990), for example, suggests but cannot explore the possibility that efficiency concerns may ex-

We, instead, consider a Mirrlees's (1971) environment — i.e., an economy inhabited by a continuum of individuals with identical preferences defined over consumption and work. Individuals differ with respect to their labor market productivity, w , which is private information. The question we pose in this environment is whether welfare losses due to the informational structure of the problem are minimal when taxes are based on the equal sacrifice principle.

Let $T(\cdot)$ be an income tax schedule derived under the equal sacrifice principle. Associated with this schedule is an equilibrium utility profile $v_1(\cdot)$, where $v_1(w)$ is the utility attained by an individual with productivity w . We ask whether there is an alternative tax schedule that generates at least as much revenue and which induces a utility profile $v^*(\cdot)$ such that $v^*(w) \geq v_1(w)$, $\forall w$, with strict inequality for a subset of positive measure of individuals.

The first step toward our goal is to derive the minimum equal sacrifice allocation: i.e., an incentive compatible allocation which generates excess resources that are sufficient to finance the government consumption needs while imposing an identical utility loss on all individuals. We use a truthful direct mechanism, to find the incentive compatible equal sacrifice allocation. This is the same approach used by Berliant and Gouveia (1993), which, to the best of our knowledge, was the first work to explicitly take into account labor supply responses in an equal sacrifice based tax problem.⁵ From the allocation we recover the equal sacrifice schedule, $T(\cdot)$ using the taxation principle. Finally, we check whether this schedule satisfies an efficiency condition for equal sacrifice schedules which we derive using Werning's (2007b) approach.

Throughout the paper, we adopt a separable iso-elastic specification for preferences. Separability is important not only for us to apply Werning's (2007a) methodology but also because, as we shall see, under separability, taxable income is invariant to the level of sacrifice. As a consequence, it is only through reduced consumption, i.e., through total taxes paid, that sacrifice is imposed on agents. This invariance of taxable income with respect to the level of sacrifice rationalizes the abstraction from labor supply responses in the early literature, thus allowing us to assess efficiency of schedules derived in these classic works.

We find that inefficiency crucially depends on the shape of economy's distribution of

plain the poor fit of equal sacrifice schedules at the high end of the distribution of income. In his words (Young (1990) p.264) "For high incomes, therefore, the departure from equal sacrifice may be due to efficiency considerations while for low income it is probably due to revenue requirements."

⁵Although Berliant and Gouveia (1993) raise the issue of efficiency, they do not address it formally. Indeed, while declaring that "One of the aspects of the model we still need to clarify are its welfare properties" and suggesting that inefficiency should result since "The condition of a zero marginal tax rate at the top ability level, emphasized in Sadka (1976) and Seade (1977), is not generally satisfied." Berliant and Gouveia (1993) never produce a systematic discussion of the issue.

skills, mirroring some results in the optimal taxation literature. For commonly used parametric distributions of skills, regions of inefficiencies for the equal sacrifice schedule, are intervals of the form $[y_a, \infty)$, with y_a representing the lowest level of income for which the marginal tax rate is 'too high'. For empirically relevant levels of government consumption, inefficiency only arises at very high levels of income and counter factually high marginal tax rates.

We finally devise a procedure to extract the marginal social welfare weights associated with the equilibrium allocations, e.g., Diamond and Mirrlees (1971a,b); Saez and Stantcheva (2013). We relate these weights to the Lagrange multipliers of the Pareto problem from which Werning's (2007b) efficiency bounds are derived and show how to express them as a function of observed variables only. This allows an immediate connection between Werning's (2007b) efficiency bounds and Bourguignon and Spadaro's (2012) revealed social preferences methodologies for assessing efficiency.

The fact that we are able to find closed form expressions for the marginal social welfare weights associated with equal sacrifice schedules allows us to ask whether these depend on variables that are easily connected to fairness principles. The dependence of marginal social weights on parameters of the distribution of skills suggests otherwise, thus posing significant challenges to the general application of these weights for policy evaluation.⁶

The rest of the paper is organized as follows. Section 2 describes the economy. Implementable allocations are described in Section 3. In Section 3.1 we derive the shape of equal sacrifice schedules for different parameters of risk aversion. The main results of this paper are found in Sections 4, 5, where efficiency bounds are derived and applied to real world situations and 6, where marginal social weights are calculated. Section 7 concludes. The appendix gathers the derivation of some of the main results.

2 The Environment

The economy is inhabited by a continuum of measure one of individuals with identical preferences defined over consumption, c , and effort, l . Preferences are represented by

$$U(c, l) = u(c) - h(l), \quad (1)$$

where u and h are smooth functions such that $u', -u'', h', h'' > 0$, $\lim_{c \rightarrow 0} u'(c) = \infty$ and $\lim_{l \rightarrow \infty} h(l) = \infty$.

⁶This is not an issue with all uses of marginal social welfare weights. Marginal weights are, of course, all that is needed for 'small' reforms. Our concern here is with the idea that one may try to '...directly augment the social welfare function to include other concerns that the society might have.' — Saez and Stantcheva (2013).

In most of what follows, we restrict our analysis to an iso-elastic specification for preferences,

$$u(c) = \frac{c^{1-\rho} - 1}{1-\rho},$$

for $\rho > 0, \rho \neq 1, u(c) = \ln c$ for $\rho = 1$ and

$$h(l) = l^\gamma / \gamma$$

for $\gamma > 1$. We do so not only for tractability but also because when preferences are iso-elastic equal sacrifice tax schedules have appealing properties, e.g. invariance with respect to rescaling — see Young (1987), Young (1988) —, a property that will facilitate our efficiency assessments.

Individuals differ from one another with respect to labor market productivity, $w \in W \subset R_+$, where W is a closed interval. We assume that w is distributed according to $F(w)$ with no mass points and associated density f , such that $f(w) > 0$ for all $w \in W$.

An individual with productivity w that makes effort l produces output $y = lw$ with y measured in units of the consumption good. Technology is, in this sense, very simple: one unit of output, y , is converted one for one into one unit of consumption, c . The economy is competitive so that each individual is paid his or her output. We, thus, refer to y as output and taxable income, interchangeably. As it turns, it is convenient to define choices over (c, y) -bundles instead of (c, l) -bundles, noting that identical preferences over (c, l) , $U(c, l) = u(c) - h(l)$, induce type-dependent preferences over (c, y) , $\tilde{U}(c, y; w) = u(c) - h(y/w)$.

Following Mirrlees (1971), we assume that w is private information. That is, neither w nor l are observed separately.

In the economy there is also a government that must extract an exogenously given amount of resources, B , from the economy.

We, therefore, define an *environment*, $\mathcal{E}$, as a triple (U, F, B) where $U(c, l) = u(c) - h(l)$, $F(\cdot)$ is the distribution of skills, and B the Government's revenue requirement, along with its informational structure.

An allocation is a mapping $(c, y) : W \rightarrow R_+^2$ that associates to each type, w , a consumption/output pair $(c(w), y(w))$. Let $\Gamma(w)$ denote the set of choices available (budget sets) for an agent of productivity w . Each $\Gamma(\cdot)$ induces an allocation, (c, y) , through

$$(c(w), y(w)) \in \operatorname{argmax}_{(c, y) \in \Gamma(w)} \{u(c) - h(y/w)\}$$

for all w .

The *no-sacrifice allocation*, in particular, is the allocation that results from $\Gamma(w) = \Gamma_0 \equiv \{(c, y); c \leq y\} \forall w$,

$$(c_0(w), y_0(w)) \equiv \operatorname{argmax}_{(c, y) \in \Gamma_0} \{u(c) - h(y/w)\}.$$

We write $v_0(w) = u(c_0(w)) - h(y_0(w)/w)$ to denote the utility attained by type w in the no-sacrifice world.⁷

This defines the economy's resource constraint,

$$B \leq \int [y(w) - c(w)]f(w)dw. \quad (2)$$

To try and induce an allocation satisfying (2), the Government designs a tax schedule defined as a mapping $T : R_+ \rightarrow R_+$ from output, y , to taxes, $T(y)$. Associated to each tax schedule is a budget set $\Gamma_T \equiv \{(y, c) \in R_+^2; c \leq y - T(y)\}$. Under this tax schedule the maximum utility attained by an individual with productivity w is

$$v_T(w) \equiv \max_y \{u(y - T(y)) - h(y/w)\}.$$

The *sacrifice induced by the tax schedule on an individual of productivity w* , $s(w)$, by

$$s(w) \equiv v_0(w) - v_T(w),$$

where $v_0(w)$ is the utility attained by type w individual when the budget set is the one associated with a chosen reference point. In all that follows we take as a reference point the 'no-sacrifice world', for which $\Gamma_0 \equiv \{(y, c) \in R_+^2; c \leq y\}$. It is apparent that the specific representation for preferences that we use defines the standard by which sacrifice is to be measured.

Equal sacrifice tax schedules are those that induce $s(w)$ constant in w ; $s(w) = s \forall w$.

3 Incentive-compatible equal-sacrifice systems.

The first step in our study is to find, for a given environment (U, F, B) , the associated equal sacrifice tax schedule. Working directly with budget sets lead us to an intractable

⁷ No-sacrifice should not be confused with a no-Government world. It may be the very presence of a Government that allows individuals to make the best use of their productivity, the latter interpretation would introduce the possibility of differentiated 'benefits' from taxation, and bring us closer to the 'benefit doctrine' as a principle of distributive justice instead of the 'ability to pay doctrine' to which the equal sacrifice principle is associated. Instead, we think of the no-sacrifice world as a counter-factual one in which all Government activities necessary to sustain the economic activity are carried on without costs. See Musgrave (1985) for an exposition.

problem. Hence, we consider a direct mechanism and use the taxation principle to recover the associated tax schedule.⁸

In thus proceeding we follow Berliant and Gouveia (1993), which, in turn, relies on Mirrlees (1971).

The Direct Mechanism We rely on the revelation principle to restrict ourselves to considering a truthful mechanism in which the planner asks each individual his or her type, w , and uses the (possibly false) report $\hat{w}$ to assign a bundle $(c(\hat{w}), y(\hat{w}))$. To guarantee truthful revelation an allocation $(c, y) = (c(w), y(w))_{w \in W}$ must be such that, for all w ,

$$w \in \operatorname{argmax}_{\hat{w} \in W} \{u(c(\hat{w})) - h(y(\hat{w})/w)\}. \quad (3)$$

Define $v(w) \equiv u(c(w)) - h(y(w)/w)$ as the value of the solution, $(c(w), y(w))$, of the problem above. The global incentive compatibility condition (3) is satisfied if and only if the envelope condition,

$$v'_1(w) = h' \left(\frac{y(w)}{w} \right) \frac{y(w)}{w^2}, \quad (4)$$

and the monotonicity condition,

$$y(w) \text{ increasing in } w, \quad (5)$$

are satisfied.

Under the assumption that $h(\cdot)$ is strictly increasing and strictly convex, (4) and (5) lead to

$$\frac{y(w)}{w} = \varphi(v'(w)w), \quad (6)$$

where φ is a strictly increasing function.

Nowhere have we used the level of utility, only its variation. This is an interesting consequence of incentive compatibility when preferences are separable: the cross-sectional *variation* in utility pins down the *level* of output produced by all individuals.

Equal sacrifice, on the other hand, restricts $(c(\cdot), y(\cdot))$ to be such that $v_0(w) - v(w) = s \forall w$, and s , to be the minimum s for which $\int [y(w) - c(w)] f(w) dw \geq B$. Note that differentiability of v_0 , implies differentiability of v and $v'_0(w) = v'(w) \forall w$. Since equal sacrifice is all about preserving 'utility differences', the cross-sectional variation of utility is invariant to the level of sacrifice.

Combining incentive compatibility, (6), and equal sacrifice, $v'_0(w) = v'(w) \forall w$, we have

⁸In this environment the equivalence between a direct mechanism and the indirect mechanism represented by the tax schedule is easy to verify — see Guesnerie (1998).

$y_1(w) = y_0(w)$ for all w . Individuals must produce the exact same output they produce at the reference state!⁹

Because everyone makes the same effort and produces the same output as in the reference state, sacrifice is all due to reduced consumption

$$s = u(y_0(w)) - u(y_0(w) - T(y_0(w))). \quad (7)$$

Inducing Invariance To understand how this non-linear budget set induces labor supply choices that are invariant to the level of sacrifice, let $I(w) = R'(y(w))y(w) - R(y(w))$, be the virtual income as defined in Hausman (1985). The linear approximation of an individual's budget constraint around his or her optimal choice is $c(w) \leq y(w)r(w) + I(w)$. By applying the definition of virtual income to expression (9), we get

$$I(w) = y(w) \left\{ r(w)^{1/\rho} - r(w) \right\}.$$

Assume that the elasticity of labor supply is negative. A decrease in net wage induced by an increase in the marginal tax rate would cause an increase in effort if taxes were linear. For a non-linear schedule, however, an decrease in $r(w)$ is associated with an increase in virtual income given that the term in curly brackets is a decreasing function of $r(w)$. Virtual income, therefore, introduces an additional income effect that adds to the traditional income effect — the one that results from the decrease in the 'price of leisure' — in such a way as to exactly offset the substitution effect. As a result, taxable income is held constant.

Note, however, that it is the elasticity of taxable income proper and not the cross-sectional one that matters for the invariance result. Because the former is not invariant to the level of sacrifice, and may even switch sign, virtual income may have to be reduced for some individuals for taxable income to be held constant — see figure 7.

3.1 The Shape of Equal Sacrifice Tax Schedules

Let $\zeta(\cdot) = u^{-1}$, then

$$T(y_0(w)) = y_0(w) - \zeta(u(y_0(w)) - s), \quad (8)$$

which allows us to recover $T(\cdot)$ using only s and $y_0(\cdot)$.

It will also be convenient to define the *retention function*, $R(y) = y - T(y)$. From (7), it is immediate that the equal sacrifice tax schedule, $T(\cdot)$, inherits the properties of $u(\cdot)$. In

⁹This result was first proven by Berliant and Gouveia (1993) — see their Proposition 4.

particular, if u is a smooth function such that $u'(x) > 0$ for all $x \in R_+$, then T'' will be a well defined continuous function.

Differentiating (7) with respect to y and rearranging terms we get¹⁰

$$\frac{u'(y)y}{u'(R(y))R(y)} = \frac{R'(y)y}{R(y)},$$

In the case of equal sacrifice schedules and constant relative risk aversion utility of consumption, marginal and average tax rates are connected through

$$\frac{R(y_0(w))}{y_0(w)} = R'(y_0(w))^{1/\rho}. \quad (9)$$

Let $r(w) = R'(y(w))$ be the *marginal retention rate* corresponding to the taxable income chosen under this schedule by an individual of productivity w . If, $\rho \neq 1$, then for a given level of sacrifice, s ,

$$r(w) = \frac{(y_0(w)^{1-\rho} - s(1-\rho))^{\frac{\rho}{1-\rho}}}{y_0(w)^\rho}, \quad (10)$$

It is immediate to verify that if, $\rho > 1$, then $\lim_{w \rightarrow \infty} r(w) = 0$.

It is also apparent from (10) that s aggregates all the information necessary to define $r(w)$. As for s , given an environment, $\mathcal{E}$, the consumption of an individual of productivity w is $c(w) = r(w) = \xi(u(y_0(w)) - s)$ if sacrifice is s .

Because $y(w)$ is invariant to s , one links the revenue raised by the Government, as a percentage of GDP, to each level of sacrifice, s , through

$$\varrho(s) = \frac{\int \{y_0(w) - \xi(u(y_0(w)) - s)\} f(w) dw}{\int y_0(w) f(w) dw}. \quad (11)$$

Finally, since $\varrho(\cdot)$ is strictly increasing in s , one finds the minimum equal sacrifice allocation by choosing s such that $\varrho(s) = b$. The associated retention schedule is given by (8).

4 Efficient Tax Schedules

Let $(c(w), y(w))$ be the allocation induced by $T(\cdot)$, and $v(w)$ the associated utility profile. A tax schedule, $T(\cdot)$ is *efficient* if and only if there is no alternative tax schedule that

¹⁰Since $R'(y) \leq R(y)/y$ is both necessary and sufficient for average taxes to be increasing, when the tax schedule is smooth, one immediately connects risk aversion and progressivity: an equal sacrifice schedule is progressive if and only if the coefficient of relative risk aversion of the chosen utility function is greater than one. Samuelson (1947) derived this result disregarding incentives and assuming that utility depended only on consumption. For the special case of separable preferences, taxable income is invariant to the level of sacrifice and Samuelson's (1947) result remains valid.

induces an allocation $(\tilde{c}(w), \tilde{y}(w))$ such that $\tilde{v}(w) \geq v(w)$ for all w and which raises more revenue.

Werning (2007b) writes down and solves this problem by replacing the incentive compatibility constraints by the traditional envelope condition. Because equal sacrifice schedules induce a taxable income function, $y(w)$, which is a strictly increasing function of w , the envelope condition is both necessary and sufficient for the allocation to be incentive compatible, in our setting. Hence, the main assumption under which Werning's (2007b) procedure is valid holds here. We combine Werning's (2007b) Proposition 4 with the properties of equal sacrifice schedules under separable and iso-elastic preferences to obtain Proposition 1, below.

Proposition 1. *For separable and iso-elastic preferences, an equal sacrifice labor income tax schedule $T(\cdot)$ is efficient if and only if marginal retention rates $r(w)$ are such that, for all w ,*

$$\eta \rho r(w)^{2-1/\rho} - [\gamma + \alpha(w) - 1] r(w) + \alpha(w) \leq \frac{\eta}{\gamma} [\gamma - 1 + \rho(\gamma + 1)], \quad (12)$$

with $\eta = \gamma/(\rho + \gamma - 1)$, and $\alpha(w) = -d \ln f / d \ln w$.

Proof. See Appendix A.1. □

Although the polynomial equation in Proposition 1 admits a closed form solution for but a few cases of interest, e.g., $\rho = 1/2$, $\rho = 2$ and $\rho = 1$,¹¹ quite a bit can be learnt from a simple exploration of (12). The left hand side of (12) is increasing in $\alpha(w)$, holding all other variables constant. Besides, (12) is satisfied for any $r(w) > 0$ if $\alpha(w) < (1 - \rho + \gamma(\rho + 1))/(\gamma + \rho - 1)$. That is, for low values of $\alpha(w)$, any positive marginal tax rate below 100% is efficient. Conversely, when $\rho > 1$ the following corollary of Proposition 1 guarantees that inefficiency always arises if the right tail of the distribution of skills is not too thick.

Corollary 1. *Let $\hat{\alpha} = \limsup \alpha(w)$, defined over the extended real numbers, and assume that $\rho > 1$. Then, if*

$$\hat{\alpha} > (1 - \rho + \gamma(\rho + 1))/(\gamma + \rho - 1), \quad (13)$$

the equal sacrifice income tax schedule exhibits inefficiencies.

Proof. Just recall that for $\rho > 1$, $\lim_{w \rightarrow \infty} r(w) = 0$. Then, for r sufficiently close to zero, (12) is violated if (13) is satisfied. □

¹¹For $\rho = 2$ a third degree polynomial defines the regions of efficiency. In the other two cases linear expressions obtain — see Appendix A.1.

A particular case of interest is when $\alpha(w)$ converges to some constant α . If this constant satisfies the inequality in Corollary 1, then an equal sacrifice schedule will eventually display inefficiencies. From a practical perspective the interesting question is in this case whether these inefficiencies arise at empirically relevant income levels. This will be the main theme of our study in Section 5.

Before, however, we present an alternative formulation for the problem studied here due to Bourguignon and Spadaro (2012).¹²

4.1 An Alternative Formulation

Let $\mathcal{U}$ be the range of U , and $\Psi : \mathcal{U} \rightarrow \mathbb{R}$ an arbitrary Bergson-Samuelson social welfare function. Then, define *the Mirrlees problem* for environment $\mathcal{E}$ given Ψ as

$$\max \int \Psi(u(c(w)) - h(y(w)/w)) f(w) dw \quad (14)$$

subject to

$$\int \{y(w) - c(w)\} f(w) dw \geq B \quad (15)$$

and

$$w \in \operatorname{argmax}_{\hat{w}} \{u(c(\hat{w})) - h(y(\hat{w})/w)\}. \quad (16)$$

We say that a tax schedule, $T(\cdot)$, is *rationalizable at environment* $\mathcal{E}$ if there is a social welfare function Ψ increasing in $v(w)$ such that the allocation that solves the Mirrlees problem at environment $\mathcal{E}$ for the social welfare function Ψ is induced by $T(\cdot)$. We then say that *the pair* $(\mathcal{E}, \Psi)$ *rationalizes* $T(\cdot)$.

The question we ask in this paper is, therefore, whether, for a given environment, $\mathcal{E}$, and an equal sacrifice tax schedule, $T(\cdot)$, is it always the case that we may find a Paretian social welfare function Ψ such that the pair $(\mathcal{E}, \Psi)$ that rationalizes $T(\cdot)$.

A related question is whether for any tax schedule $T(\cdot)$ derived under the equal sacrifice principle it is possible to find a pair $(\mathcal{E}, \Psi)$ that rationalizes $T(\cdot)$. For this second question we are given the degree of freedom to choose the environment as well as the social welfare function.

Proposition 2 in Werning (2007b) states that "For any tax schedule $T(y)$ and its resulting allocation, there is a set of skill distributions $F(w)$ and net endowments $-B$ for which the outcome is Pareto Efficient and another set of skill distributions $F(w)$ and net endowments, $-B$, for which it is Pareto inefficient." It is clear then that we may always build an

¹²Also related is the use of *generalized social marginal welfare weights* championed by Saez and Stantcheva (2013).

environment for which the tax schedule derived under the equal sacrifice principle is rationalizable. Interestingly, the same proposition states that for any tax schedule it is always possible to find an environment for which the tax schedule is *not* rationalizable. Indeed, as we have seen from Corollary 1 all that we need is to assume that the distribution of skills satisfies (13) for inefficiency to arise.¹³ The degree of freedom one gets if one is allowed to choose the environment is sufficient to get this ‘anything goes’ result.

In appendix A.2 we show how to use Werning’s (2007b) procedure to extract the marginal social weights — see Diamond and Mirrlees (1971a,b); Saez (2001); Saez and Stantcheva (2013) — associated with the equal sacrifice schedule. In our numeric exercises we apply this alternative procedure as well. Regions for which marginal weights become negative are exactly those for which (12) is violated.

5 Sacrifice and Efficiency in Practice

We are interested in applying Proposition 1 to real world economies. Toward this end we need to define values for the parameters γ and ρ along with estimates of $\alpha(w)$ for all w and a level of sacrifice, s , that are empirically relevant.

Let us start with the parameters γ and ρ . The most obvious way of finding sensible values for this parameter is to find a pair of statistics that we are interesting in matching using (ρ, γ) . For example, we may try and match both the elasticity and the compensated elasticity of taxable income using ρ and γ . In this case, the two parameters will be uniquely pinned down.

By choosing both parameters, we commit to a specific perception of sacrifice. That is, to benefit from the separable specification of utility, we have renounced the possibility of playing around with the utility representation while still holding preferences fixed. An alternative is to use a different procedure, for which, we start with a society’s perception of fairness, which we capture through ρ . Given ρ , γ is chosen in order to match the cross-sectional elasticity of taxable income, $\epsilon = d \ln y / d \ln w$ (which in our case is $\eta = \gamma / (\gamma + \rho - 1)$).¹⁴ By doing so, we refrain from matching other parameters that we may find relevant; e.g., compensated and income elasticities.

In Section 5.1, we use both approaches. First, we use both the elasticity and the compensated elasticity of taxable income to simultaneously obtain ρ and γ . In doing so, we refraining from exploring different social perception of sacrifice, ρ . Next, we vary the perception of sacrifice by varying ρ and match only the elasticity of taxable income. In this

¹³Of course, this is but a consequence of Kaplow and Shavell’s (2001) result.

¹⁴We refer to the elasticity of taxable income with respect to w , $d \ln y / d \ln w$. Some studies define it, instead, as $d \ln y / d \ln(1 - \tau)$. We shall return to this point at the end of Section 5.1.

case, we use Young's (1990) empirical assessment of ρ as the basis for the perception of sacrifice. The compensated elasticity is, then, a result of these two choices. As it turns, for the values for the two elasticities we consider, we obtain a value for ρ within the range estimated by Young (1990) to be the relevant range for which ρ generates the cross-sectional relationship between gross and net income observed in the data.

Next, to understand how we may determine $d \ln f / d \ln w$ consistently with our choices for the other parameters, let

$$\hat{y}(w) = \operatorname{argmax}_y \{u(y - \hat{T}(y)) - h(y/w)\},$$

where $\hat{T}(\cdot)$ is the actual income tax schedule of the economy we study. Assume that, given our chosen parameters and the actual tax schedule, $\hat{T}(\cdot)$, the function $\hat{y}(\cdot)$ has an inverse, i.e., there is $\omega(\cdot)$ such that $w = \omega(\hat{y}(w))$.^{15,16} Using the observed distribution of taxable income, $G(y)$, we find the distribution of w , $F(w)$ through $G(y) = F(\omega(y))$.

Finally, we determine s , using (11).

The $u(\cdot) = \ln(\cdot)$ case. Before studying the US economy, let us consider a very simple example that allows for simple back of the envelope calculations and helps us illustrate the efficiency test.

Let $u(\cdot) = \ln(\cdot)$, and assume that the actual tax schedule of the economy we are interested in studying can be approximated by a linear one. Also assume that this linear tax system induces a Pareto distribution of income, $G(y) = 1 - (\underline{y}/y)^{\varphi-1}$, with support $[\underline{y}, \infty)$, $\underline{y} > 0$, and associated density $g(y) = \kappa y^{-\varphi}$, $\varphi > 1$, where $\kappa = (\varphi - 1)\underline{y}^{\varphi-1}$.

Under these assumptions $y(w) = w$, and $\omega(y) = y$. Therefore,¹⁷

$$\alpha(w) = -\frac{d \ln f}{d \ln w} = -\frac{d \ln g(y)}{d \ln y} = \varphi.$$

This is a commonly used specification for the distribution of income — e.g., Diamond

¹⁵In the case of progressive piece-wise linear taxes $y_T(w)$ is not invertible. There is bunching of individuals at the concave kinks. It is not possible, in this case to identify F along the interval of bunching. In practice, we do not observe bunching at the concave kinks, which creates a difficulty for one to map the simple model presented here to the data.

¹⁶This rules out concave kinks in the actual tax schedule. For $\hat{y}(w)$ to be a function, rather than a correspondence, we rule out convex kinks, as well. As it turns, this is less of a problem for our procedure: by defining $\hat{y}(w)$ as a selection from $\hat{y}(w) \in \operatorname{argmax}_y \{u(y - T(y)) - h(y/w)\}$, the procedure works as before.

¹⁷The distribution of skills that generates the distribution of income $G(y)$ is, therefore,

$$F(w) = 1 - \left(\frac{w}{\underline{w}}\right)^{(\alpha-1)},$$

with $\alpha = \varphi$, for $w \in [\underline{w}, \infty)$.

(1998); Saez (2001); Werning (2007b) — which allows us to borrow the value of the key parameter α from the literature.

If $\alpha < 2$ the condition in Proposition 1 is satisfied for any value of r . When $\alpha > 2$, using $\rho = 1$, $\alpha(w) = \alpha$, and $\eta = 1$ in (12) we get

$$r(w) = r \geq \frac{\alpha - 2}{\alpha + \gamma - 2}, \quad (17)$$

which allows for simple back of the envelope calculations.

Saez (2001) considers the following values of φ for the US economy: 2.5, 3 and 3.5. Werning (2007b), $\varphi = 3$.¹⁸ In all cases, condition (17) has a bite.

As for the other parameters, recall that we may no longer use ϵ to determine γ . ϵ is identically zero for all γ : income and substitution effects exactly offset. Yet, $\epsilon = 0$ is compatible with compensated and income elasticities being both low or both high. Which is the case is exactly what γ determines. Thus, we may choose γ to match the compensated (or the income) elasticity of taxable income. Alternatively, we may simply recall that $\epsilon^f = \gamma/(\gamma - 1)$ is the Frisch elasticity of taxable income, for which reliable estimates are available.

With linear taxes, $b = 1 - r = \tau$. We, then, report our findings as the maximum level revenue requirements as a percentage of GDP that may be raised without violating (17).

Assume that $\varphi = 2.5$. When $\gamma = 2$, $\epsilon^f = 1$ expenditures must be at least 80% of GDP for inefficiency to result, whereas for $\gamma = 4$ Government consumption of up to 89% of GDP may be efficiently financed under the equal sacrifice principle. $\gamma = 4$ is a sensible number for it implies $\epsilon^f = 0.33$, which is closer to the lower end of the numbers used in the macro literature. This value for γ is, moreover, associated with a compensated elasticity of labor supply of 0.25 in line with the empirical evidence. Now, even if we take $\epsilon^f \rightarrow \infty$, the maximum value for b is still 67% of GDP.

When $\varphi = 3$, the preferred value for both Saez (2001) and Werning (2007b), inefficiency arises for lower levels of Government consumption. Depending on the value of ϵ^f , the bound for the marginal tax rate will be 80% for $\epsilon^f = 0.33$, 67% for $\epsilon^f = 1$, and will approach 50% as $\epsilon^f \rightarrow \infty$. Finally, if $\varphi = 3.5$, the values for the bounds are 73% for $\epsilon^f = 0.33$, 57% for $\epsilon^f = 1$, and 40% as $\epsilon^f \rightarrow \infty$.

If one is willing to accept that $u(\cdot) = \ln(\cdot)$ reasonably describes the way the American society perceives ability to pay, expenditures as a share of GDP need, therefore, to be too high, when compared to what one observes in the US for inefficiencies to arise. This message should, however, be taken with caution. As we shall see, the use of a Pareto

¹⁸Recall that a Pareto distribution does not have a finite mean if $\alpha < 2$, and it does not have a finite variance if $\alpha < 3$.

distribution plays a significant role for these findings.

Although interesting from illustrative purposes, $\rho = 1$, reflects but one possible perception of sacrifice. In what follows, we consider $\rho > 1$, recalling that the evidence in Young (1990) suggests that a value for ρ in the range $[1.5, 1.7]$ is needed for the equal sacrifice principle to rationalize the US tax schedule for the 1957-1987 period. Moreover, as previously explained, $\rho > 1$ is needed for equal sacrifice tax schedules to be progressive.

5.1 Main results for the US economy

Given (ρ, γ) , we follow the procedure described in Section 5 to retrieve F from the data, using the actual US tax schedule, $T(\cdot)$, and distribution of income, $G(y)$. To guarantee that $\hat{y}(w)$ may be inverted and to simplify the procedure, we follow Saez (2001) in assuming that the actual tax system may be reasonably approximated by a linear one, $R(y) = ry$.¹⁹ Under $R(y) = ry \forall y$,

$$\omega(y) = y^{\frac{\rho+\gamma-1}{\gamma}} r^{\frac{1-\rho}{\gamma}}.$$

Moreover, $y_0(w) = \hat{y}(w) r^{\frac{1-\rho}{\gamma+\rho-1}}$.

Figure 3 displays the distribution $F(\cdot)$ recovered from labor income data from the Panel Study of Income Dynamic (PSID) for the year 2007 using $\rho = 1.5$ and $\gamma = 1.5$. These parametric choices yield a cross-sectional elasticity of taxable income of .75 (and a corresponding elasticity of labor supply of $-.25$), which is in the low end of the range we shall explore. A Generalized Extreme Value distribution is adjusted to the empirical distribution of skills to generate a parametric $F(\cdot)$ with a smooth function $\alpha(w)$.

If the density is single peaked with $\alpha(w)$ decreasing in w , after $\ln f$ reaches its maximum, there is a subset of the skills interval where inefficiency can never arise, since $\alpha(w) < 0 < (1 - \rho + \gamma(\rho + 1))/(\gamma + \rho - 1)$ prior to the modal value of f . We may, in this case disregard all the ascending part of the skills density, if we know the level of sacrifice, s . Put somewhat differently, for single peaked densities the information pertaining to the ascending part of f ($\alpha(w) < 0$) is only used to determine s .

If, in addition, $\rho > 1$ and $\alpha(w)$ is non-decreasing, as most of the commonly approximated parametric distributions of skills are, a region of inefficiency, may it exist, must be an interval of the form $[w_a, \infty)$, where w_a is the lowest level of productivity for which the marginal tax rate is too high.²⁰

¹⁹Note that, by adopting a linear approximation for the current schedule we either assume that the current system cannot be rationalized by an equal sacrifice one or that a $\ln$ specification reflects the current perception of sacrifice. Either view is in contrast with Young's (1990) findings for the US in the 1957-1987.

²⁰An important example of such properties arise if a Pareto density approximates well the descending part of the skills density function — Diamond (1998); Saez (2001); Werning (2007b) — in which case $\alpha(w) = -d \ln f / d \ln w = \alpha, \forall w$. Equally relevant is the case of log-normal distributions — Mirrlees (1971).

In our first exercise we assume that the underlying distribution of skills is log-normal. We chose $\rho = 1.5$, and $\gamma = 3.5$, corresponding to cross-sectional elasticities of labor supply and compensated elasticities of labor supply of $-.125$, and $.25$, respectively. The level of sacrifice is chosen to raise 30% of GDP in tax revenues.

For illustrative purposes, instead of the polynomial equation 12, Figure 4 displays $r(\cdot)$ and $\bar{r}(\cdot)$ as defined in (22) as a function of taxable income, y . Because, $\rho > 1$, the marginal tax rate is increasing in y . For the log-normal distribution, $\alpha(w)$ is non-decreasing and the region of inefficiency is an $[w_a, \infty)$. The region of inefficiency only arises for annual income above US\$ 895k, even though the log-normal distribution has values for $\alpha(w)$ which are excessively high in this region. In the same figure we illustrate the possibility of inefficiencies by requiring the government to raise 30% of GDP in tax revenues. In this case, inefficiency arises when the marginal retention rate reaches 39% and taxable income is close to US\$ 607k.

The use of a log-normal distribution allows us to match very closely the median, first and third quartiles and even the 9th decile of the distribution. For the top percentiles, however, a Pareto distribution offers a better fit. We, then adjust a Pareto distribution to the top of the distribution of skills.

In figure 5, we display the lower bounds for marginal retention rates both for the log-normal and the Pareto distributions. Note, however that for both cases we use the same level of sacrifice, chosen in such a way as to raise 20% of GDP, using the log-normal distribution. For higher levels of income, the use of a log-normal distribution causes inefficiency to arise earlier. Indeed, for α as high as 3.5 the difference between the bounds of a Pareto and a log-normal distribution are still significant, as the figure clearly shows. Inefficiency only arises for the Pareto distribution when taxable income reaches US\$ 3,272k. Even if revenues are 30% of GDP, inefficiency only arises when $r = 23\%$ and the taxable income is US\$ 1,159k.

The contrasting results for a log-normal distribution and a Pareto distribution are hardly surprising. Indeed, they mirror the current debate on the efficient taxation of high earners.²¹

Young's (1990) equal sacrifice schedules Below, we report the critical value of $r(\cdot)$ and the associated critical level of income for each value of ρ used in Young's (1990) work. We also show how much revenue would be raised for this level of sacrifice if the distribution of skills was the one we have extracted from current data. This provides a rough

²¹Diamond (1998) was probably the first to emphasize how the results regarding optimal taxes may dramatically change when one substitutes a Pareto distribution for a log-normal one (initially used by Mirrlees (1971)).

estimate of whether these are compatible with actual Government consumption values. In our exercises, we vary γ to hold the elasticity of labor supply constant at -1.25 .

For 1957, Young (1990) found that a value of $\rho = 1.610$ provided the best fit for the data. Using the log-normal distribution of skills that we adjusted to the US, we find that inefficiency arises when $r = 30\%$ which takes place for one whose yearly earnings are US\$194k. Tax revenue reaches 37% of GDP, for the level of sacrifice estimated by Young.

When $\rho = 1.519$, corresponding to the year of 1967, we find that the inefficiency arises when $r = 39\%$ and yearly earnings are US\$250k. Tax revenues are a little over 28% of GDP.

Finally, when $\rho = 1.72$ (and $\gamma = 3.3$), corresponding to the year of 1977, inefficiency arises at $r = 33\%$ for a taxable income of US\$230k. Tax revenues are 28% of GDP.

In all these cases, the level of revenue as a percentage of GDP is well above Government consumption of any country in peace time. This findings are not surprising if one considers that taxes may have been used to finance some redistribution. The trouble with this finding is that no redistribution takes place in the equal sacrifice problem described by Young (1990).

Next, taking into account the fact that the level of sacrifice that we used is borrowed from Young (1990) we need not use the entire distribution of skills to apply Proposition 1 if $\alpha(w)$ is weakly increasing. We then assume that the upper part of the distribution of skills is well approximated by a Pareto distribution. Because the Pareto distribution only matches the top of the distribution we refrain from making any statement regarding total revenues raised.

For $\rho = 1.519$ (and $\gamma = 3.5$), we find critical values of $r = 39\%$ and $y = \text{US\$1,220k}$. For $\rho = 1.610$ (and $\gamma = 3.4$), $r = 16\%$ and $y = \text{US\$486k}$. Finally, for $\rho = 1.718$ (and $\gamma = 3.3$), we obtain $r = 16\%$ and $y = \text{US\$692k}$. In all exercises we used $\alpha = 3.5$.

6 Marginal Social Weights

As we have discussed in Section 4.1, an alternative procedure for assessing efficiency is to try and recover the implicit marginal welfare weights that justifies a society's choice of tax schedule. If these are everywhere positive, then the tax schedule is Pareto efficient.

Our procedure to recover the marginal weights is simple. We note that the marginal welfare weights are related to the Lagrange multiplier associated with the promise keeping constraint (27) in the Pareto problem that defines the efficiency bounds — see appendix A.2. It is then a matter of algebra to arrive at an expression for this Lagrange multiplier that only depends on observables.

We present two different weights. The first is simply $\Psi'(v(w))$. This is the direct

(marginal) placed on one's utility. It shows how, at the margin, *utility* differences affects social welfare. It has the natural property of being constant for a utilitarian model. Of course this is not the sole reason for why a society may weight differently *income* in different hands. The curvature of the utility function plays a role here. We also display, $\Psi'(v(w))c(w)^{-\rho}$, which is what Diamond and Mirrlees (1971a,b) call the *social marginal utility of income* in the hands of individual type w . Saez and Stantcheva (2013) call the same term, the *generalized social marginal welfare weight*. They call it generalized because if it is allowed to depend on non-welfarist considerations as well. In both cases we normalize weights to integrate to one.

Proposition 2. *Let $\underline{w} > 0$ be the lowest skill level of the distribution of skills. The social marginal utility of income in the hands of a type- w person is*

$$\Psi'(\bar{v}(w))c(w)^{-\rho} = \lambda \left\{ Y(w) \frac{1 - r(w)}{\gamma r(w)} + 1 \right\},$$

where,

$$Y(w) \equiv -\Phi(w) - \frac{d \ln r(w)}{d \ln w} \frac{r(w)}{1 - r(w)} + \gamma,$$

and

$$\lambda = \left\{ \int c(w)^\rho f(w) dw - \frac{1}{\gamma - 1} \frac{1 - r(\underline{w})}{r(\underline{w})} f(\underline{w}) c(\underline{w})^\rho \right\}^{-1}.$$

Proof. See Appendix A.2. □

Figure 6 displays both the social marginal utility of income and the generalized social marginal welfare weights for two different levels of sacrifice, corresponding to the schedules derived for the log-normal distribution in Section 5.1.

Because we obtain a closed form solution for the marginal social welfare weights, it is possible to check which parameters affect their value. Somewhat disappointingly, variables which are hard to reconcile with any idea of fairness, e.g. $\alpha(w)$, show up in the the marginal weights expression.

Discussion and Caveats We have relied on a separable specification of preferences to keep the analysis simple. The invariance of taxable income with respect to the level of sacrifice allows one to rationalize the findings of the early literature, and better communicate the main aspects of the problem. Under this specification of preferences, however we had to choose $\rho \geq 1$ if we were to account for the progressivity of most tax modern systems. This choice implies, in turn, positive cross-sectional elasticities of taxable income which are positive.

Although many studies — e.g., Gruber and Saez (2002) — report negative values for the elasticity of taxable income, it is important to recall that under non-linear taxes, cross-sectional elasticities are not the same as proper elasticities. Figure 7, for example, displays the elasticity of taxable income for different levels of income, under a typical equal sacrifice schedule. If we take at face value the findings by Young (1990) we should see in the data, as well a large discrepancy between the cross-sectional and the elasticity proper of taxable income. Yet, it is not unlikely that the values we use are lower than the empirical ones. One possible reason why this would be the case is that the tax elision or evasion channel that may play an important role — e.g., Piketty et al. (2011) — is absent from our model.

A potential shortcoming of our analysis is that we have ruled out taxation for purely redistributive reasons by identifying the world of no-sacrifice as a world of no-taxes or transfers in which an agent's consumption is his or her output. It is possible to incorporate redistributive motives of taxes by changing the definition of a no-sacrifice world — e.g., by considering a world where a minimum level of consumption is guaranteed to all. Most of our formulae would change, but the methodology would still apply.

Note, however, that a most appealing consequence of our assumption regarding the no-sacrifice world is that the invariance property, guaranteed by separability, leads to a notion of sacrifice that is only a function of *observed* gross and net income: both observable variables. This feature alone may be the very reason why the notion of sacrifice may be easily accessible to most people. Without it, sacrifice would have to refer to a counter-factual world that may be hard for one to grasp.

7 Conclusion

Young (1988) has shown that any method of apportioning taxes that satisfies a series of sensible properties is an equal sacrifice schedule for some utility function, whereas Young (1990) has taken real world distributions of before and after tax incomes and has shown that one could find a common (and empirically sound) utility function that equalizes the utility loss of all individuals, and such that this loss was minimal to finance the government revenue requirements. Taken together these findings are suggestive that the simplicity of the notion of equal sacrifice along with the sensible properties of income taxes derived from it may have influenced the political debate and found its way into the actual design of tax schedules.

We assess efficiency of income tax schedules using a separable iso-elastic specification for preferences in a Mirrlees's (1971) setting. Separability greatly facilitates the derivation of equal sacrifice schedules and allows for an explicit evaluation of efficiency using the

methodology developed by Werning (2007b).

We vary the preference parameters in a range that produces sensible elasticities of taxable income. For the iso-elastic specification this implies a coefficient of risk aversion that varies in the interval between 1.5 and 1.7, which according to Young (1990) is also what is needed to rationalize the US income tax schedule using the equal sacrifice principle.

Using a log-normal distribution of skills, and our preferred parametrization of preferences, for an equal sacrifice schedule to finance 20% of the US GDP, inefficiencies only arise when marginal tax rates reach 58% at an income level of US\$ 895k. If, instead, the top of the US distribution of skills is approximated by a Pareto distribution then inefficiencies only arise for an income level of US\$3,274k, and a marginal tax rate of 72% for a decay parameter as high as 3.5.

Finally, we extract the social marginal weights associated with the tax schedules generated by the equal sacrifice principle. We find their informational content to be very limited for one to use it for any non-local policy prescription.

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A Appendix

A.1 Proofs

Lemma 1. For all $s > 0, 0 < r < 1$.

Proof. Let $u_0(w) = u(c_0(w))$ and use the fact that utility differences are the same for all w to see that

$$\frac{\xi'(u_0(w) - s)}{\xi'(u_0(w))} = r(w),$$

where $\xi'(u)$ is the marginal cost, measured in consumption units, of delivering utility u and $r(w) = R'(y(w))$. Note that ξ is an increasing convex function of u which means that $0 < \tau < 1$ for $u(y_0) - \underline{u} > s > 0$, where $\underline{u} = \lim_{c \rightarrow 0} u(c)$. $\square$

Proof of Proposition 1.

Proof. Let $R(\cdot)$ be a smooth retention function and $y(w)$ the induced efficient labor income schedule. Assume that the associated marginal retention function $r(\cdot)$, is $r(w) > 0 \forall w$. Differentiating (7) and rearranging terms yields

$$\frac{u'(y_0(w))}{u'(R(y_0(w)))} = R'(y_0(w)). \quad (18)$$

Differentiate (18) to obtain

$$\begin{aligned} \frac{R''(y_0(w))y_0(w)}{R'(y_0(w))} &= y_0(w) \left\{ \frac{u''(R(y_0(w)))}{u'(R(y_0(w)))} R'(y_0(w)) - \frac{u''(y_0(w))}{u'(y_0(w))} \right\} \\ &= \left\{ A^r(y_0(w)) - A^r(R(y_0(w))) \frac{R'(y_0(w))}{R(y_0(w))} y_0(w) \right\}, \end{aligned} \quad (19)$$

where $A^r(y)$ is the coefficient of relative risk aversion at income level y . For the case of CRRA preferences, $A^r(c) = \rho$ for all c , and expression (19) reduces to

$$-\frac{d \ln R'(y)}{d \ln y} \Big|_{y=y_0(w)} = \rho \left\{ 1 - \frac{R'(y_0(w))}{R(y_0(w))} y_0(w) \right\}.$$

Recall that $r(w) = R'(y(w))$ and note that

$$\frac{r'(w)}{r(w)} w = \frac{R''(y(w))}{R'(y(w))} y(w) \frac{y'(w)}{y(w)} w$$

or,

$$\begin{aligned}\frac{d \ln r(w)}{d \ln w} &= \frac{d \ln R'(y(w))}{d \ln y(w)} \frac{d \ln y(w)}{d \ln w} \\ &= \rho \left\{ r(w)^{1-\frac{1}{\rho}} - 1 \right\} \frac{d \ln y(w)}{d \ln w}\end{aligned}$$

where we have used (9) in the last step. Using $\eta = \gamma / (\gamma + \rho - 1)$, we have

$$\frac{d \ln r(w)}{d \ln w} = \left\{ r(w)^{1-\frac{1}{\rho}} - 1 \right\} \rho \eta. \quad (20)$$

Next, note that the equal sacrifice schedule induces a taxable income profile, $y(\cdot)$, that is increasing in w . The first order conditions used by Werning (2007b) to derive his results are therefore also sufficient for the allocation to be implementable. Define²²

$$\Phi(w) \equiv -\frac{d \ln f(w)}{d \ln w} + (\gamma - 1) \frac{d \ln y(w)}{d \ln w} - 1. \quad (21)$$

Assume that $\Phi(w) > d \ln r(w) / d \ln w$, then, Werning's (2007b) Proposition 4 adapted to our setting states that a retention function $R : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is efficient if and only if

$$r(w) \geq \frac{\Phi(w) - \gamma}{\Phi(w) - \frac{d \ln r(w)}{d \ln w}}. \quad (22)$$

To save on notation, let $\alpha(w) = -d \ln f(w) / d \ln w$. Then, for a equal sacrifice retention function associated with separable iso-elastic preferences we have

$$\Phi(w) = (\gamma - 1) \frac{\gamma}{\gamma + \rho - 1} + \alpha(w) - 1. \quad (23)$$

Substituting (23) in (22), and using (20) we get

$$r(w) [\gamma + \alpha(w) - 1] - r(w)^{2-1/\rho} \frac{\rho \gamma}{\gamma + \rho - 1} \geq \alpha(w) - \frac{\gamma - 1 + \rho(1 + \gamma)}{\gamma + \rho - 1}, \quad (24)$$

for all w . □

Condition (22) need not always have a bite. We have assumed that $\Phi(w) > d \ln r(w) / d \ln w$, which guarantees that the denominator in (22) is positive. If, however $\Phi(w) < \gamma$, all one

²²The term $d \ln y / d \ln w$ that appears in equation (21) is the *cross-sectional elasticity of taxable income*, i.e. the percentage change in taxable income when we compare individuals whose productivities differ by one percent for a given tax structure. This makes the application of (22) quite simple under the separable iso-elastic specification for preferences, since $d \ln y / d \ln w = \gamma / (\gamma + \rho - 1)$ for all levels of sacrifice. The elasticity of taxable income proper, in contrast, is *not* invariant to the level of sacrifice. See figure 7.

is requiring is for the marginal retention rate to be positive, or, equivalently, for marginal tax rates not to exceed 100 per cent. On the other hand, if $\Phi(w) > d \ln r(w)/d \ln w$, and $d \ln r(w)/d \ln w > \gamma$, then only with marginal retention rates that exceed 100 per cent will the schedule be efficient.²³

A.2 Marginal Social Welfare Weights

Deriving Condition (22) Next, we show how the procedure devised in Werning (2007b) can be used to extract the marginal social welfare weights, in the sense of Saez and Stantcheva (2013). Note that we shall only provide a sketch of proof of necessity for efficiency condition (22). A complete proof of both necessity and sufficiency is found in Werning (2007b).

Following Werning (2007b), if a tax schedule is efficient the underlying allocation must solve the following program:

$$\max_{y(\cdot), v(\cdot)} \int [y(w) - e(v(w), y(w), w)] f(w) dw$$

s.t.,

$$v'(w) = \frac{y(w)^\gamma}{w^{\gamma+1}}, \quad (25)$$

$$y(w) \text{ increasing}, \quad (26)$$

and

$$v(w) \geq \bar{v}(w) \forall w. \quad (27)$$

Disregarding the monotonicity constraint (25), we may write the Lagrangian

$$\int \left\{ [y(w) - e(v(w), y(w), w)] f(w) + \mu(w) \left[v'(w) - \frac{y(w)^\gamma}{w^{\gamma+1}} \right] + \lambda(w) [v(w) - \bar{v}(w)] \right\} dw,$$

where

$$v(w) = u(e(v(w), y(w), w)) - \frac{y(w)^\gamma}{\gamma w^\gamma}.$$

²³ $d \ln r(w)/d \ln w$ is positive (resp., negative) if marginal tax rates are increasing (resp., decreasing). If $d \ln r(w)/d \ln w > 0$, then $\Phi(w) > 0$ suffices. Note also that in the derivation of () we have assumed $0 < r(w) < 1$, which is always the case in our model. A reversed inequality results when $r(w) > 1$ using a similar derivation which does not rely on taking logs.

Integrating by parts

$$\int \left\{ [y(w) - e(v(w), y(w), w)] f(w) - \mu'(w)v(w) - \mu(w) \frac{y(w)^\gamma}{w^{\gamma+1}} + \lambda(w) [v(w) - \bar{v}(w)] dw \right\} + \mu(\bar{w})v(\bar{w}) - \mu(\underline{w})v(\underline{w})$$

First order conditions are,

$$(1 - e_y(v(w), y(w), w)) f(w) = \mu(w) \gamma \frac{y(w)^{\gamma-1}}{w^{\gamma+1}}, \quad (28)$$

$$-e_v(v(w), y(w), w) f(w) = \mu'(w) - \lambda(w), \quad (29)$$

$\mu(\underline{w}) \leq 0, \lim_{w \rightarrow \infty} \mu(w) = 0$ ($\mu(\bar{w}) \geq 0$, if F has a bounded support).

This implies,

$$-e_v(v(w), y(w), w) f(w) \leq \mu'(w). \quad (30)$$

To save on notation, let $\bar{e}_v(w) \equiv e_v(v(w), y(w), w)$, and $\bar{e}_y(w) \equiv e_y(v(w), y(w), w)$. Then, assuming $1 > \bar{e}_y(w) = r(w) > 0$, (28) can be written in logs,

$$\ln(1 - r(w)) + \ln f(w) = \ln \mu + \ln \gamma + (\gamma - 1) \ln y(w) - (\gamma + 1) \ln w,$$

which implies

$$\frac{d \ln(1 - r(w))}{d \ln w} + \frac{d \ln f(w)}{d \ln w} = \frac{d \ln \mu(w)}{d \ln w} + (\gamma - 1) \frac{d \ln y(w)}{d \ln w} - (\gamma + 1). \quad (31)$$

Next note that

$$\begin{aligned} \frac{d \ln \mu(w)}{d \ln w} = \frac{\mu'(w)}{\mu(w)} w &\geq -\frac{\bar{e}_v(w) f(w)}{\mu(w)} w = -\frac{\gamma y(w)^{\gamma-1} \bar{e}_v(w) f(w) w}{(1 - \bar{e}_y(w)) f(w) w^{\gamma+1}} = \\ &= -\frac{\gamma}{1 - r(w)} \bar{e}_v(w) \frac{y(w)^{\gamma-1}}{w^\gamma} = -\gamma \frac{r(w)}{1 - r(w)}. \end{aligned}$$

Hence, using (31), we get

$$-\frac{d \ln r(w)}{d \ln w} \frac{r(w)}{1 - r(w)} + \frac{d \ln f(w)}{d \ln w} \geq -\gamma \frac{r(w)}{1 - r(w)} + (\gamma - 1) \frac{d \ln y(w)}{d \ln w} - (\gamma + 1).$$

We may, then, write the efficiency condition for $r(w)$ as

$$r(w) \left[\Phi(w) - \frac{d \ln r(w)}{d \ln w} \right] \geq \Phi(w) - \gamma,$$

where $\Phi(w)$ is as defined in (21). Assuming $\Phi(w) > d \ln r(w) / d \ln w$, equation (22) obtains.

Proof of Proposition 2

Proof. Using the same definitions applied throughout the paper, the problem defined in Section 4.1—henceforth, primal program—may be written,

$$\max \int \Psi(v(w)) f(w) dw \quad (32)$$

s.t.,

$$\int [y(w) - e(v(w), y(w), w)] f(w) dw \geq G, \quad (33)$$

$$v'(w) = h' \left(\frac{y(w)}{w} \right) \frac{1}{w}, \quad (34)$$

and $y(w)$ increasing.

The associated Lagrangian is

$$\begin{aligned} & \int \{ \Psi(v(w)) + \lambda [y(w) - e(v(w), y(w), w) - G] \} f(w) dw \\ & - \int \left\{ \bar{\mu}'(w) v(w) + \bar{\mu}(w) h' \left(\frac{y(w)}{w} \right) \frac{1}{w} \right\} dw \end{aligned}$$

with transversality conditions, $\bar{\mu}(\underline{w}) \leq 0$ and $\lim_{w \rightarrow \infty} \bar{\mu}(w) = 0$.

The first order conditions are

$$\{ \Psi'(v(w)) - \lambda e_v(v(w), y(w), w) \} f(w) = \bar{\mu}'(w), \quad (35)$$

and

$$\lambda [1 - e_y(v(w), y(w), w)] f(w) = \bar{\mu}(w) h'' \left(\frac{y(w)}{w} \right) \frac{y(w)}{w^2}. \quad (36)$$

Let $(\bar{v}(w), \bar{y}(w))$ denote the allocation that solves the primal program above.

Comparing (29) with (35), and (28) with (36) it is apparent that $(\bar{v}(w), \bar{y}(w))$ solves (29) and (28) for

$$\zeta(w) = \frac{\Psi'(\bar{v}(w)) f(w)}{\lambda},$$

and

$$\frac{\bar{\mu}(w)}{\lambda} = \mu(w).$$

Integrating (35) and applying the transversality conditions,

$$\int \{ \Psi'(\bar{v}(w)) - \lambda e_v(\bar{v}(w), \bar{y}(w), w) \} f(w) dw = \int \bar{\mu}'(w) dw = -\bar{\mu}(\underline{w}),$$

which leads to

$$\lambda = \frac{\int \Psi'(\bar{v}(w)) f(w) dw + \bar{\mu}(\underline{w})}{\int e_v(\bar{v}(w), \bar{y}(w), w) f(w) dw}$$

Next, note that

$$\lambda [1 - e_y(v(\underline{w}), y(\underline{w}), \underline{w})] f(\underline{w}) = \bar{\mu}(\underline{w}) h'' \left(\frac{y(\underline{w})}{\underline{w}} \right) \frac{y(\underline{w})}{\underline{w}^2},$$

which allows one to write

$$\lambda \int e_v(\bar{v}(w), \bar{y}(w), w) f(w) dw - \int \Psi'(\bar{v}(w)) f(w) dw = \bar{\mu}(\underline{w})$$

Adopting the normalization $\int \Psi'(\bar{v}(w)) f(w) dw = 1$, and after some straightforward though tedious algebra, we may re-write () as

$$\lambda = \left\{ \int e_v(\bar{v}(w), \bar{y}(w), w) f(w) dw - \frac{1 - r(\underline{w})}{(\gamma - 1)r(\underline{w})} f(\underline{w}) e_v(\bar{v}(\underline{w}), \bar{y}(\underline{w}), \underline{w}) \right\}^{-1},$$

where we have used

$$\begin{aligned} h''(n(\underline{w})) \frac{n(\underline{w})}{\underline{w}} &= \frac{h''(n(\underline{w})) n(\underline{w})}{h'(n(\underline{w}))} \frac{h'(n(\underline{w}))}{\underline{w}} \frac{e_v(\bar{v}(\underline{w}), \bar{y}(\underline{w}), \underline{w})}{e_v(\bar{v}(\underline{w}), \bar{y}(\underline{w}), \underline{w})} \\ &= \frac{(\gamma - 1)r(\underline{w})}{e_v(\bar{v}(\underline{w}), \bar{y}(\underline{w}), \underline{w})}. \end{aligned}$$

When $\bar{\mu}(\underline{w}) = 0$, $r(\underline{w}) = 1$ and $\lambda = \{ \int e_v(\bar{v}(w), \bar{y}(w), w) f(w) dw \}^{-1}$. For our purposes, however, equal sacrifice will always lead to $\mu(\underline{w}) = \bar{\mu}(\underline{w}) / \lambda > 0$.

In any case we may recover, from the data, using

$$\zeta(w) = \frac{\Psi'(\bar{v}(w)) f(w)}{\lambda},$$

provided that we know $\mu'(w)$.

The only issue is how to find an expression for $\mu'(w)$. For this we may use (28), which we re-write simply as

$$[1 - r(w)] f(w) = \mu(w) \gamma y(w)^{\gamma-1} w^{-\gamma-1}. \quad (37)$$

Differentiating (37) leads to

$$\begin{aligned} [1 - r(w)] f'(w) - r'(w) f(w) &= \mu'(w) \gamma \frac{y(w)^{\gamma-1}}{w^{\gamma+1}} - \mu(w) \gamma (\gamma + 1) \frac{y(w)^{\gamma-1}}{w^{\gamma+2}} \\ &\quad + \mu(w) \gamma (\gamma - 1) \frac{y(w)^{\gamma-2}}{w^{\gamma+1}} y'(w), \end{aligned}$$

which may also be written,

$$Y(w) [1 - r(w)] f(w) = \mu'(w) \gamma \frac{y(w)^{\gamma-1}}{w^\gamma},$$

where,²⁴

$$Y(w) \equiv \frac{f'(w)w}{f(w)} - \frac{r'(w)w}{[1 - r(w)]} - \left\{ (\gamma - 1) \frac{y'(w)w}{y(w)} - (\gamma + 1) \right\}.$$

Then, we obtain the following expression for $\mu'(w)$,

$$\mu'(w) = Y(w) \frac{1 - r(w)}{\gamma r(w)} e_v(\bar{v}(w), \bar{y}(w), w) f(w),$$

which gives us

$$\zeta(w) = \left\{ Y(w) \frac{1 - r(w)}{\gamma r(w)} + 1 \right\} e_v(v(w), y(w), w) f(w).$$

Finally,

$$\Psi'(\bar{v}(w)) = \lambda \left\{ Y(w) \frac{1 - r(w)}{\gamma r(w)} + 1 \right\} e_v(v(w), y(w), w).$$

□

B Figures

²⁴Note that $Y(w)$ can equivalently be written,

$$Y(w) \equiv -\Phi(w) - \frac{d \ln r(w)}{d \ln w} \frac{r(w)}{1 - r(w)} + \gamma.$$

Figure 1: Invariance Property: The figure displays an equal sacrifice schedule for iso-elastic preferences as well as optimal choices both at the reference point—indifference curves are straight lines—and under the equal sacrifice schedule—indifference curves are dotted lines. We consider individuals of two different productivities: $w=120$ and $w=150$.

Figure 2: The two top figures display marginal and average tax rates as a function of taxable income, for three different levels of sacrifice. In all figures, solid line refers to the lowest level of sacrifice and dotted line to the highest. The bottom left figure displays Virtual Income defined as $(1 - \tau)y - T(y)$. The bottom right figure shows how the elasticity of taxable income $d \ln y / d \ln(1 - \tau)$ varies with taxable income.

Figure 3: The figure displays the distribution of skills we obtain by using PSID data for the year 2007, and approximating the tax system by a linear one with $\tau = .3$. We also adjust a Generalized Extreme Value distribution to the empirical distribution.

Figure 4: This figure displays the marginal retention rates, r , associated with each level of taxable income as well as the minimum marginal retention rate, $\bar{r}(\cdot)$, for which the tax system is efficient. It is assumed that the distribution of skills is log-normal, $\rho = 1.5$, $\gamma = 3.5$, and two levels of tax revenues are considered: 20% - low G -, and 30% - high G - of GDP.

Figure 5: This figure displays the marginal retention rate, r , associated with each level of taxable income as well as the minimum marginal retention rate for which the tax system is efficient. Both a log-normal and a Pareto distribution of skills, with $\alpha = 3.5$ are considered. The preference parameters are $\rho = 1.5$, $\gamma = 3.5$, and the level of sacrifice is chosen to generate tax revenues of 20% of GDP for the log-normal distribution.

Figure 6: The left panel displays marginal social welfare weights, $\Psi'(\cdot)$, for two different levels of revenue as a percentage of GDP, 20% - low G, and 30% - high G. In the right panel the associated marginal social values of income, $\Psi'(\cdot)c(\cdot)^{-\rho}$, are displayed. Both are normalized to integrate to one.

Figure 7: The left column compares marginal retention rates, bounds (top row), marginal social welfare weights (middle row) and marginal value of income (bottom row) for two economies that have the same gini coefficient but different mean income. The right column compares the same things for two economies with the same mean income but different gini coefficients.